

# Special weak Dirichlet processes and BSDEs driven by a random measure

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## Abstract

This paper considers a forward BSDE driven by a random measure, when the underlying forward process  $X$  is special semimartingale, or even more generally, a special weak Dirichlet process. Given a solution  $(Y, Z, U)$ , generally  $Y$  appears to be of the type  $u(t, X_t)$  where  $u$  is a deterministic function. In this paper we identify  $Z$  and  $U$  in terms of  $u$  applying stochastic calculus with respect to weak Dirichlet processes.

**Key words:** Weak Dirichlet processes; Calculus via regularizations; Random measure; Stochastic integrals for jump processes; Backward stochastic differential equations.

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## 1 Introduction

This paper considers a forward BSDE driven by a random measure, when the underlying forward process  $X$  is special semimartingale, or even more generally, a special weak Dirichlet process. Given a solution  $(Y, Z, U)$ , often  $Y$  appears to be of the type  $u(t, X_t)$  where  $u$  is a deterministic function. In this paper we identify  $Z$  and  $U$  in terms of  $u$  applying stochastic calculus with respect to weak Dirichlet processes.

Indeed the employed techniques perform the calculus with respect to (special) weak Dirichlet processes developed in [2]. In that paper we also extend the stochastic calculus via regularizations to the case of jump processes. Given some filtration  $(\mathcal{F}_t)$ , we recall that a special weak Dirichlet process is a process of the type  $X = M + A$ , where  $M$  is an  $(\mathcal{F}_t)$ -local martingale and  $A$  is an  $(\mathcal{F}_t)$ -predictable orthogonal process, see Definition 2.7. When  $A$  has bounded variation, then  $X$  is a special  $(\mathcal{F}_t)$ -semimartingale. The decomposition of a special weak Dirichlet process is unique, see Proposition 2.12. A significant result of [2] is the chain rule stated in Theorem 2.14, concerning the expansion of  $F(t, X_t)$ , where  $X$  is a special weak Dirichlet process of finite quadratic variation and  $F$  is of class  $C^{0,1}$ . If we know a priori that  $F(t, X_t)$  is the sum of a bounded variation process and a continuous  $(\mathcal{F}_t)$ -orthogonal process, then the chain rule only requires  $F$  to be continuous; in that case no assumptions are required on the càdlàg process  $X$ .

As we have already mentioned, we will focus on forward BSDEs, which constitute a particular case of BSDEs in its general form. BSDEs have been deeply studied since the seminal paper [24]

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by Pardoux and Peng. In [24], as well as in many subsequent papers, the standard Brownian motion is the driving process (Brownian context) and the concept of BSDE is based on a non-linear martingale representation theorem with respect to the corresponding Brownian filtration. A recent monograph on the subject is [26]. BSDEs driven by processes with jumps have also been investigated: two classes of such equations appear in the literature. The first one relates to BSDEs where the Brownian motion is replaced by a general càdlàg martingale  $M$ , see, among others, [4], [14], [6]. An alternative version of BSDEs with a discontinuous driving term is the one associated to an integer-valued random measure  $\mu$ , with corresponding compensator  $\nu$ . In this case the BSDE is driven by a continuous martingale  $M$  and a compensated random measure  $\mu - \nu$ . In that equation naturally appears a purely discontinuous martingale which is a stochastic integral with respect to  $\mu - \nu$ , see, e.g., [33], [5], [32]. A recent monograph on BSDEs driven by Poisson random measures is [13].

In this paper we will focus on BSDEs driven by random measures (we will use the one-dimensional formalism for simplicity). Besides  $\mu$  and  $\nu$  appear three driving random elements: a continuous martingale  $M$ , a non-decreasing adapted continuous process  $\zeta$  and a predictable random measure  $\lambda$  on  $\Omega \times [0, T] \times \mathbb{R}$ , equipped with the usual product  $\sigma$ -fields. Given a square integrable random variable  $\xi$ , and two measurable functions  $\tilde{g} : \Omega \times [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\tilde{f} : \Omega \times [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ , the equation takes the following form:

$$\begin{aligned} Y_t = \xi &+ \int_{]t, T]} \tilde{g}(s, Y_{s-}, Z_s) d\zeta_s + \int_{]t, T] \times \mathbb{R}} \tilde{f}(s, e, Y_{s-}, U_s(e)) \lambda(ds de) \\ &- \int_{]t, T]} Z_s dM_s - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de). \end{aligned} \quad (1.1)$$

As we have anticipated before, the unknown of (1.1) is a triplet  $(Y, Z, U)$  where  $Y, Z$  are adapted and  $U$  is a predictable random field. The Brownian context of Pardoux-Peng appears as a particular case, setting  $\mu = \lambda = 0$ ,  $\zeta_s \equiv s$ . There  $M$  is a standard Brownian motion and  $\xi$  is measurable with respect to the Brownian  $\sigma$ -field at terminal time. In that case the unknown can be reduced to  $(Y, Z)$ , since  $U$  can be arbitrarily chosen. Another significant subcase of (1.1) arises when only the purely discontinuous driving term appears, i.e.  $M$  and  $\zeta$  vanish; under this simpler structure the related BSDE can be approached by an iterative method: a significant example is represented by BSDEs driven by a marked point process, as in [9]. Connections between the martingale and the random measure driven BSDEs are illustrated by [22].

When the random dependence of  $\tilde{f}$  and  $\tilde{g}$  is provided by a Markov solution  $X$  of a forward SDE, and  $\xi$  is a real function of  $X$  at the terminal time  $T$ , then the BSDE (1.1) is called forward BSDE, the one that we have anticipated at the beginning. This generally constitutes a stochastic representation of a partial integro-differential equation (PIDE). In the Brownian case, when  $X$  is the solution of a classical SDE with diffusion coefficient  $\sigma$ , then the PIDE reduces to a semilinear parabolic PDE. If  $v : [0, T] \times \mathbb{R} \times \mathbb{R}$  is a classical (smooth) solution of the mentioned PDE, then  $Y_s = v(s, X_s)$ ,  $Z_s = \sigma(s, X_s) \partial_x v(s, X_s)$ , generate a solution to the forward BSDE, see e.g. [27], [25], [28]. In the general case when the forward BSDEs are also driven by random measures, similar results have been established, for instance by [3], for the jump-diffusion case, and by [8], for the purely discontinuous case, i.e. when no Brownian noise appears. In the context of martingale driven forward BSDEs, a first approach to the probabilistic representation has been carried on in [23].

Conversely, solutions of forward BSDEs generate solutions of PIDEs in the viscosity sense. More precisely, for each given couple  $(t, x) \in [0, T] \times \mathbb{R}$ , consider an underlying process  $X$  given by the solution  $X^{t,x}$  of an SDE starting at  $x$  at time  $t$ . Let  $(Y^{t,x}, Z^{t,x}, U^{t,x})$  be a family of solutions of the forward BSDE. In that case, under reasonable general assumptions, the function  $v(t, x) := Y_t^{t,x}$

is a viscosity solution of the related PIDE. A demanding task consists in characterizing the couple  $(Z, U) := (Z^{t,x}, U^{t,x})$ , in term of  $v$ ; this is generally called the *identification problem* of  $(Z, U)$ . In the continuous case, this was for instance the object of [17]: the authors show that if  $v \in C^{0,1}$ , then  $Z_s = \partial_x v(s, X_s)$ ; under more general assumptions, they also associate  $Z$  with a generalized gradient of  $v$ . At our knowledge, in the discontinuous case, the problem of the identification of the martingale integrands couple  $(Z, U)$  has not been deeply investigated, except for particular situations, as for instance the one treated in [9]: this problem was faced in [8].

In the present paper we discuss the mentioned identification problem in a quite general framework by means of the calculus related to weak Dirichlet processes. When  $Y$  is a deterministic function  $v$  of a special semimartingale  $X$ , related in a specific way to the random measure  $\mu$ , we apply the chain rule in Theorem 2.14 in order to identify the couple  $(Z, U)$ . This is the object of Proposition 4.12. The result remains valid if  $X$  is a special weak Dirichlet process with finite quadratic variation. In the purely discontinuous framework, i.e. when in the BSDE (1.1)  $M$  and  $\zeta$  vanish, we make use of the chain rule Theorem 2.16, which, for a general càdlàg process  $X$ , allows to express  $v(t, X_t)$  without requiring any differentiability on  $v$ . In particular Theorem 2.16 does not ask  $X$  to be a special weak Dirichlet process, provided we have some a priori information on the structure of  $v(t, X_t)$ . The identification in that case is stated in Proposition 4.18. We remark that in most of the literature on BSDEs, the measure  $\nu, \lambda, \zeta$  of equation (1.1) are non-atomic in time. A challenging case arises when one or more of those predictable processes have jumps in time. Well-posedness of BSDEs in that case has been partially discussed in [1] in the purely discontinuous case, and in a slightly different context by [7], for BSDEs driven by a countable sequence of square-integrable martingales. Our approach to the identification problem also applies to forward BSDEs presenting predictable jumps.

The paper is organized as follows. In Section 2 we fix the notations and we recall some important results about the calculus related to weak Dirichlet processes with jumps developed in [2]. In Section 3 we introduce a class of stochastic processes  $X$  related in a specific way to a given integer-valued random measure  $\mu$ , and we provide some technical results on related stochastic integration. Section 4 is devoted to solve the identification problem. In Appendix A we recall some useful results related to the general theory of stochastic processes. Finally, in Appendix B we report some basic results on the stochastic integration with respect to random measures; in particular we emphasize the connection between integer-valued random measures and the jumps of a càdlàg process.

## 2 Preliminaries

In what follows, we are given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  a positive horizon  $T$  and a filtration  $(\mathcal{F}_t)_{t \geq 0}$ , satisfying the usual conditions. Let  $\mathcal{F} = \mathcal{F}_T$ . Given a topological space  $E$ , in the sequel  $\mathcal{B}(E)$  will denote the Borel  $\sigma$ -field associated with  $E$ .  $\mathcal{P}$  (resp.  $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ ) will denote the predictable  $\sigma$ -field on  $\Omega \times [0, T]$  (resp. on  $\tilde{\Omega} = \Omega \times [0, T] \times \mathbb{R}$ ). Analogously, we set  $\mathcal{O}$  (resp.  $\tilde{\mathcal{O}} = \mathcal{O} \otimes \mathcal{B}(\mathbb{R})$ ) as the optional  $\sigma$ -field on  $\Omega \times [0, T]$  (resp. on  $\tilde{\Omega}$ ). Moreover,  $\tilde{\mathcal{F}}$  will be  $\sigma$ -field  $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R})$ , and we will indicate by  $\mathcal{F}^{\mathbb{P}}$  the completion of  $\mathcal{F}$  with the  $\mathbb{P}$ -null sets. We set  $\tilde{\mathcal{F}}^{\mathbb{P}} = \mathcal{F}^{\mathbb{P}} \otimes \mathcal{B}([0, T] \times \mathbb{R})$ . By default, all the stochastic processes will be considered with parameter  $t \in [0, T]$ . The symbols  $\mathbb{D}^{ucp}$  and  $\mathbb{L}^{ucp}$  will denote the space of adapted càdlàg and càglàd processes endowed with the u.c.p. (uniform convergence in probability) topology. By convention, any càdlàg process defined on  $[0, T]$  is extended to  $\mathbb{R}_+$  by continuity.

A bounded variation process  $X$  on  $[0, T]$  will be said to be with integrable variation if the expectation of its total variation is finite.  $\mathcal{A}$  (resp.  $\mathcal{A}_{\text{loc}}$ ) will denote the collection of all adapted processes with integrable variation (resp. with locally integrable variation), and by  $\mathcal{A}^+$  (resp.  $\mathcal{A}_{\text{loc}}^+$ )

the collection of all adapted integrable increasing (resp. adapted locally integrable) processes. The significance of locally is the usual one which refers to localization by stopping times, see e.g. (0.39) of [20].

We will indicate by  $C^{1,2}$  (resp.  $C^{0,1}$ ) the space of all functions

$$u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \quad (t, x) \mapsto u(t, x)$$

that are continuous together their derivatives  $\partial_t u$ ,  $\partial_x u$ ,  $\partial_{xx} u$  (resp.  $\partial_x u$ ).  $C^{1,2}$  is equipped with the topology of uniform convergence on each compact of  $u$ ,  $\partial_x u$ ,  $\partial_{xx} u$ ,  $\partial_t u$ .  $C^{0,1}$  is equipped with the same topology on each compact of  $u$  and  $\partial_x u$ .

## 2.1 Càdlàg processes and the associated jump measures

The concept of random measure allows a very tractable description of the jumps of a càdlàg process, and will be extensively used throughout the paper. For the convenience of the reader we have summarized in Appendix A and B respectively the concepts on general theory of stochastic processes and on integer-valued random measures we will need in the following, for more details consult Chapter I and Chapter II, Section 1, in [21], and Chapter XI, Section 1, in [18].

Given a càdlàg process  $X = (X_t)_{t \in [0, T]}$ , its *jump measure* is the integer-valued random measure

$$\mu^X(\omega; dt dx) = \sum_{s \in ]0, T]} \mathbb{1}_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt dx). \quad (2.1)$$

The compensator of  $\mu^X$  is called the Lévy system of  $X$ , and will be denoted by  $\nu^X$ . The jump measure  $\mu^X$  acts in the following way: for any positive  $\tilde{\mathcal{O}}$ -measurable function  $W$  we have

$$\sum_{s \in ]0, T]} \mathbb{1}_{\{\Delta X_s \neq 0\}} W_s(\cdot, \Delta X_s) = \int_{]0, T] \times E} W_s(\cdot, x) \mu^X(\cdot, ds dx).$$

In the sequel we will make often use of the following assumption on the processes  $X$ :

$$\sum_{s \in ]0, T]} |\Delta X_s|^2 < \infty, \quad \text{a.s.} \quad (2.2)$$

*Remark 2.1.* Condition (2.2) holds for instance in the case of processes  $X$  of finite quadratic variation.

The two propositions below were the object of Propositions 2.4 and 2.7 in [2].

**Proposition 2.2.** *Let  $p = 1, 2$ . Let  $X$  be a real-valued càdlàg process on  $[0, T]$  such that*

$$\sum_{s \in ]0, T]} |\Delta X_s|^p < \infty, \quad \text{a.s.} \quad (2.3)$$

*Then*

$$\int_{]0, \cdot] \times \mathbb{R}} |x|^p \mathbb{1}_{\{|x| \leq 1\}} \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}^+. \quad (2.4)$$

**Proposition 2.3.** *Let  $X$  be a càdlàg process on  $[0, T]$  satisfying condition (2.2), and let  $F$  be a function of class  $C^{0,1}$ . Then*

$$\int_{]0, \cdot] \times \mathbb{R}} |(F(s, X_{s-} + x) - F(s, X_{s-}))|^2 \mathbb{1}_{\{|x| \leq 1\}} \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}, \quad (2.5)$$

$$\int_{]0, \cdot] \times \mathbb{R}} |x \partial_x F(s, X_{s-})|^2 \mathbb{1}_{\{|x| \leq 1\}} \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}. \quad (2.6)$$

In particular, the stochastic integrals

$$\begin{aligned} & \int_{]0, \cdot] \times \mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-})) \mathbb{1}_{\{|x| \leq 1\}} (\mu^X - \nu^X)(ds dx), \\ & \int_{]0, \cdot] \times \mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} (\mu^X - \nu^X)(ds dx) \end{aligned}$$

define two purely discontinuous square integrable local martingales.

The following condition on  $X$  will play a fundamental role in the sequel:

$$\int_{]0, \cdot] \times \mathbb{R}} |x| \mathbb{1}_{\{|x| > 1\}} \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}^+. \quad (2.7)$$

*Remark 2.4.* (a) Condition (2.7) holds for instance if  $X$  is a special semimartingale, see Corollary 11.26 in [18].

(b) If  $X$  is a special semimartingale satisfying  $\sum_{s \in ]0, T]} |\Delta X_s| < \infty$  a.s., from point (a) and Proposition 2.2 with  $p = 1$ , we have

$$\int_{]0, \cdot] \times \mathbb{R}} |x| \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}^+. \quad (2.8)$$

We will be interested in functions  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  fulfilling the integrability property

$$\int_{]0, \cdot] \times \mathbb{R}} |F(s, X_{s-} + x) - F(s, X_{s-}) - x \partial_x F(s, X_{s-})| \mathbb{1}_{\{|x| > 1\}} \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}^+. \quad (2.9)$$

*Remark 2.5.* (i) Condition (2.9) is automatically verified if  $X$  is a càdlàg process satisfying (2.7), and  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a function of  $C^{0,1}$  class with  $\partial_x F$  bounded.

(ii) If  $X$  is a càdlàg process satisfying condition (2.7), and  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a function of class  $C^{0,1}$  fulfilling (2.9), then

$$\int_{]0, \cdot] \times \mathbb{R}} |F(t, X_{t-} + x) - F(t, X_{t-})| \mathbb{1}_{\{|x| > 1\}} \mu^X(dt dx) \in \mathcal{A}_{\text{loc}}^+,$$

see Lemma 5.21 in [2].

*Remark 2.6.* Let  $\varphi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a  $\tilde{\mathcal{P}}$ -measurable function and  $A$  a  $\tilde{\mathcal{P}}$ -measurable subset of  $\Omega \times [0, T] \times \mathbb{R}$ , such that

$$|\varphi| \mathbb{1}_A * \mu^X \in \mathcal{A}_{\text{loc}}^+, \quad (2.10)$$

$$|\varphi|^2 \mathbb{1}_{A^c} * \mu^X \in \mathcal{A}_{\text{loc}}^+. \quad (2.11)$$

Then the process  $\varphi$  belongs to  $\mathcal{G}_{\text{loc}}^1(\mu^X)$ .

As a matter of fact, (2.10) and Proposition B.18 give that  $\varphi \mathbb{1}_A$  belongs to  $\mathcal{G}_{\text{loc}}^1(\mu^X)$ . On the other hand, (2.11), together with Lemma B.21-2), implies that  $\varphi \mathbb{1}_{A^c}$  belongs to  $\mathcal{G}_{\text{loc}}^2(\mu^X) \subset \mathcal{G}_{\text{loc}}^1(\mu^X)$ .

## 2.2 Recalls on calculus related to special weak Dirichlet processes

In the present section we will recall the main results of calculus related to special weak Dirichlet processes. The proofs of those results can be found in [2], to which the reader may also refer for a more complete treatise of the subject.

We start by recalling the notion of special weak Dirichlet process and some important results associated to this concept.

**Definition 2.7.** *Let  $X$  be an  $(\mathcal{F}_t)$ -adapted process. We say that  $X$  is  $(\mathcal{F}_t)$ -orthogonal if  $[X, N] = 0$  for every  $N$  continuous local  $(\mathcal{F}_t)$ -martingale.*

*Remark 2.8.* A basic example of  $(\mathcal{F}_t)$ -orthogonal processes are purely discontinuous  $(\mathcal{F}_t)$ -martingales, see Theorem 7.34 in [18] and the comments above.

**Definition 2.9.** *Let  $X$  be an  $(\mathcal{F}_t)$ -adapted process.*

- (i)  *$X$  is called  $(\mathcal{F}_t)$ -Dirichlet process if it admits a decomposition  $X = M + A$ , where  $M$  is a local martingale and  $A$  is a finite quadratic variation process with  $[A, A] = 0$ ;*
- (ii)  *$X$  is called  $(\mathcal{F}_t)$ -weak Dirichlet process if it admits a decomposition  $X = M + A$ , where  $M$  is a local martingale and the process  $A$  is  $(\mathcal{F}_t)$ -orthogonal;*
- (iii)  *$X$  is called  $(\mathcal{F}_t)$ -special weak Dirichlet process if it admits a decomposition of the type (ii) and, in addition,  $A$  is predictable.*

When the underlying filtration is clear, we will often omit the filtration, speaking in particular about Dirichlet processes, weak Dirichlet processes, special weak Dirichlet processes.

*Remark 2.10.* If  $S$  is an  $(\mathcal{F}_t)$ -semimartingale which is a special weak Dirichlet process, then it is a special semimartingale, see Proposition 5.9 in [2].

*Remark 2.11.* We observe that the validity of condition (2.7) extends to the processes of the type  $X = S + A$ , where  $S$  is a special semimartingale and  $A$  is a continuous process. This is the case for instance when  $X$  is an  $(\mathcal{F}_t)$ -Dirichlet process.

A special weak Dirichlet process admits the following unique decomposition, which was the object of Proposition 5.7 in [2].

**Proposition 2.12.** *Let  $X$  be an  $(\mathcal{F}_t)$ -special weak Dirichlet process of the type*

$$X = M^c + M^d + A, \tag{2.12}$$

*where  $M^c$  is a continuous local martingale, and  $M^d$  is a purely discontinuous local martingale. Supposing that  $A_0 = M_0^d = 0$ , the decomposition (2.12) is unique.*

*Remark 2.13.* (a) Decomposition (2.12) will be called the canonical decomposition of  $X$ .

- (b) Identity (2.12) gives the most general form of a special weak Dirichlet process, since every local martingale  $M$  can be decomposed as the sum of a continuous local martingale  $M^c$  and a purely discontinuous local martingale  $M^d$ , see Theorem A.8.

We recall below a stability results for special weak Dirichlet processes, that is crucial in the present work, which was the object of Theorem 5.26 in [2].

**Theorem 2.14.** *Let  $X$  be an  $(\mathcal{F}_t)$ -special weak Dirichlet process of finite quadratic variation with its canonical decomposition  $X = M^c + M^d + A$ . Assume that conditions (2.9) and (2.7) hold. Then, for every  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^{0,1}$ , we have*

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_x F(s, X_s) dM_s^c \\ &\quad + \int_{]0, t] \times \mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-})) (\mu^X - \nu^X)(ds dx) + A^F(t), \end{aligned} \quad (2.13)$$

where  $A^F : C^{0,1} \rightarrow \mathbb{D}^{ucp}$  is a linear map and, for every  $F \in C^{0,1}$ ,  $A^F$  is a predictable  $(\mathcal{F}_t)$ -orthogonal process.

*Remark 2.15.* We recall that (2.13) shows that  $(F(t, X_t))$  is a weak Dirichlet process and provides its canonical decomposition.

In some cases, the differentiability requirements on  $F$  stated in Theorem 2.14 will be no longer necessary. This is illustrated in Proposition 2.16 below, which was the object of Proposition 5.29 in [2].

**Proposition 2.16.** *Let  $X$  be an  $(\mathcal{F}_t)$ -adapted càdlàg process. Let  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that the following holds.*

- (i)  $F(s, X_s) = B_s + A'_s$ , where  $B$  is a bounded variation process and  $A'$  is a continuous  $(\mathcal{F}_t)$ -orthogonal process;
- (ii)  $\int_{]0, \cdot] \times \mathbb{R}} |F(s, X_{s-} + x) - F(s, X_{s-})| \mu^X(ds dx) \in \mathcal{A}_{\text{loc}}^+$ .

Then  $F(t, X_t)$  is an  $(\mathcal{F}_t)$ -special weak Dirichlet process with decomposition

$$F(t, X_t) = F(0, X_0) + \int_{]0, t] \times \mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-})) (\mu^X - \nu^X)(ds dx) + A^F(t), \quad (2.14)$$

where  $A^F$  is a predictable  $(\mathcal{F}_t)$ -orthogonal process.

### 3 A class of stochastic processes $X$ related in a specific way to an integer-valued random measure $\mu$

Let  $\mu$  be an integer-valued random measure on  $[0, T] \times \mathbb{R}$ , and  $\nu$  a "good" version of the compensator of  $\mu$ , as constructed in Proposition B.11-(c). Set

$$\begin{aligned} D &= \{(\omega, t) : \mu(\omega, \{t\} \times \mathbb{R}) > 0\}, \\ J &= \{(\omega, t) : \nu(\omega, \{t\} \times \mathbb{R}) > 0\}, \\ K &= \{(\omega, t) : \nu(\omega, \{t\} \times \mathbb{R}) = 1\}. \end{aligned}$$

*Remark 3.1.*  $D$  is a thin set,  $J$  is the predictable support of  $D$ , and  $K$  is the largest predictable subset of  $D$ , see Proposition B.6 and Theorem B.10. The definition of predictable support of a random set is recalled in Definition A.25.

We formulate now an assumption on a generic càdlàg process  $X$  which will be related in the sequel to the integer-valued random measure  $\mu$ .

**Hypothesis 3.2.**  $X = X^i + X^p$ , with  $X^i$  (resp.  $X^p$ ) a càdlàg quasi-left continuous adapted process (resp. càdlàg predictable process).



**Proposition 3.3.** *Let  $X$  be a càdlàg adapted process fulfilling Hypothesis 3.2. Then the two properties below hold.*

(i)  $\Delta X^p \mathbb{1}_{\{\Delta X^i \neq 0\}} = 0$  and  $\Delta X^i \mathbb{1}_{\{\Delta X^p \neq 0\}} = 0$ , up to an evanescent set.

(ii)  $\{\Delta X \neq 0\}$  is the disjointed union of the random sets  $\{\Delta X^p \neq 0\}$  and  $\{\Delta X^i \neq 0\}$ .

*Proof.* (i) Recalling Propositions A.17 (resp. A.19), there exist a sequence of predictable times  $(T_n^p)_n$  (resp. totally inaccessible times  $(T_n^i)_n$ ) that exhausts the jumps of  $X^p$  (resp.  $X^i$ ). On the other hand,  $\Delta X_{T_n^i}^p = 0$  a.s. for every  $n$ , see Proposition A.17 (resp.  $\Delta X_{T_n^p}^i = 0$  a.s. for every  $n$ , see Definition A.18), so that

$$\begin{aligned}\Delta X^i \mathbb{1}_{\{\Delta X^p \neq 0\}} &= \Delta X^i \mathbb{1}_{\cup_n [[T_n^p]]} = 0, \\ \Delta X^p \mathbb{1}_{\{\Delta X^i \neq 0\}} &= \Delta X^p \mathbb{1}_{\cup_n [[T_n^i]]} = 0.\end{aligned}$$

(ii) From point (i) we get

$$\begin{aligned}\{\Delta X \neq 0\} &= \{(\Delta X^i + \Delta X^p) \neq 0\} \\ &= \{(\Delta X^i \mathbb{1}_{\{\Delta X^p = 0\}} + \Delta X^p \mathbb{1}_{\{\Delta X^p \neq 0\}}) \neq 0\} \\ &= \{\Delta X^i \mathbb{1}_{\{\Delta X^p = 0\}} \neq 0\} \cup \{\Delta X^p \neq 0\} \\ &= \{\Delta X^i \neq 0\} \cup \{\Delta X^p \neq 0\}.\end{aligned}$$

□

**Proposition 3.4.** *Let  $X$  be a càdlàg adapted process satisfying Hypothesis 3.2. Then the properties below hold.*

1.  $\{(\omega, t) : \nu^X(\omega, \{t\} \times \mathbb{R}) > 0\} = \{\Delta X^p \neq 0\};$

2.  $\{\Delta X^p \neq 0\}$  is the largest predictable subset of  $\{\Delta X \neq 0\}$  (up to an evanescent set).

*Proof.* 1.  $\{\Delta X \neq 0\}$  is the support of the random measure  $\mu^X$  (see e.g. Proposition B.8). By Theorem B.10, the predictable support of  $\{\Delta X \neq 0\}$  is given by  $\{(\omega, t) : \nu^X(\{t\} \times \mathbb{R}) > 0\}$ .

On the other hand, by Proposition 3.3-(ii),  $\{\Delta X \neq 0\}$  is the disjointed union of  $\{\Delta X^p \neq 0\}$  and  $\{\Delta X^i \neq 0\}$ . Since  $X^i$  is a càdlàg quasi-left continuous process, by Proposition A.26 we know that the predictable support of  $\{\Delta X^i \neq 0\}$  is evanescent. By Definition A.25 of predictable support, taking into account the additivity of the predictable projection operator,  ${}^p(\mathbb{1}_{\{\Delta X \neq 0\}}) = \mathbb{1}_{\{\Delta X^p \neq 0\}}$ , and this concludes the proof.

2. By Proposition 3.3-(ii),

$$\{\Delta X^p \neq 0\} \subset \{\Delta X \neq 0\}. \quad (3.1)$$

Since  $\{(\omega, t) : \nu^X(\{t\} \times \mathbb{R}) = 1\}$  is the largest predictable subset of  $\{\Delta X \neq 0\}$  (see again Theorem B.10), it follows from point 1. and (3.1) that  $\{\Delta X^p \neq 0\}$  coincides with  $\{(\omega, t) : \nu^X(\{t\} \times \mathbb{R}) = 1\}$ .

□

*Remark 3.5.* We remark that item 2. in Proposition 3.4 has an interest in itself but will not be used in the sequel.

**Proposition 3.6.** *Let  $X$  satisfy Hypothesis 3.2 with decomposition  $X = X^i + X^p$ . Let moreover  $(S_n)_n$  be a sequence of predictable times exhausting the jumps of  $X^p$ . Then*

$$\nu^X(\{S_n\}, dx) = \mu^X(\{S_n\}, dx) \text{ for any } n, \text{ a.s.} \quad (3.2)$$



*Remark 3.7.* Since  $\{\Delta X^p \neq 0\}$  is a predictable thin set (see Definition A.4), the existence of a sequence of predictable times exhausting the jumps of  $X^p$  is a well-known fact, see Proposition A.17 and Definition A.1 for the definition of an exhausting sequence.

*Proof.* Let us fix  $n$  and let  $(E_m)_m$  be a sequence of measurable subsets of  $\mathbb{R}$  which is  $\pi$ -class generating  $\mathcal{B}(\mathbb{R})$ . Since  $X^i$  is a càdlàg quasi-left continuous adapted process and  $S_n$  is a predictable time, then  $\Delta X_{S_n}^i = 0$  a.s., see Definition A.18. This implies that  $\Delta X_{S_n} = \Delta X_{S_n}^p$  a.s. by Hypothesis 3.2. Consequently, for every  $m$  we have

$$\mathbb{1}_{E_m}(\Delta X_{S_n}^p) = \mathbb{1}_{E_m}(\Delta X_{S_n}) = \int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \mu^X(\{S_n\}, dx) \text{ a.s.} \quad (3.3)$$

On the other hand, by Proposition B.11-(b) and (3.3) we have

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \nu^X(\{S_n\}, dx) &= \mathbb{E} \left[ \int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \mu^X(\{S_n\}, dx) \middle| \mathcal{F}_{S_n-} \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{E_m}(\Delta X_{S_n}^p) \middle| \mathcal{F}_{S_n-} \right] \\ &= \mathbb{1}_{E_m}(\Delta X_{S_n}^p) \text{ a.s.,} \end{aligned}$$

where the latter equality follows from Corollary A.24. By (3.3), there exists a  $\mathcal{P}$ -measurable null set  $\mathcal{N}_m$  such that

$$\int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \nu^X(\{S_n\}, dx) = \int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \mu^X(\{S_n\}, dx) \text{ for every } \omega \notin \mathcal{N}_m.$$

Define  $\mathcal{N} = \cup_m \mathcal{N}_m$ , then

$$\int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \nu^X(\{S_n\}, dx) = \int_{\mathbb{R}} \mathbb{1}_{E_m}(x) \mu^X(\{S_n\}, dx) \text{ for every } m \text{ and } \omega \notin \mathcal{N}.$$

Then the claim follows by a monotone class argument, see Theorem 21, Chapter 1, in [12].  $\square$

We now recall an important notion of measure associated with  $\mu$ , given in formula (3.10) in [20].

**Definition 3.8.** Let  $(\tilde{\Omega}_n)$  be a partition of  $\tilde{\Omega}$  constituted by elements of  $\tilde{\mathcal{O}}$ .  $M_\mu^\mathbb{P}$  denotes the  $\sigma$ -finite measure on  $(\tilde{\Omega}, \tilde{\mathcal{F}}^\mathbb{P})$ , such that for every  $W : \tilde{\Omega} \rightarrow \mathbb{R}$  positive, bounded,  $\tilde{\mathcal{F}}^\mathbb{P}$ -measurable function,

$$M_\mu^\mathbb{P}(W \mathbb{1}_{\tilde{\Omega}_n}) = \mathbb{E}[W \mathbb{1}_{\tilde{\Omega}_n} * \mu_T]. \quad (3.4)$$

*Remark 3.9.* Formally speaking we have  $M_\mu^\mathbb{P}(d\omega, ds, de) = d\mathbb{P}(\omega) \mu(\omega, ds, de)$ .

In the sequel we will formulate the following assumption for a generic càdlàg process  $Y$  with respect to the random measure  $\mu$ .

**Hypothesis 3.10.**  $Y$  is a càdlàg adapted process satisfying  $\{\Delta Y \neq 0\} \subset D$ . Moreover, there exists a  $\tilde{\mathcal{P}}$ -measurable map  $\tilde{\gamma} : \Omega \times ]0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\Delta Y_t(\omega) \mathbb{1}_{]0, T]}(t) = \tilde{\gamma}(\omega, t, \cdot) \quad dM_\mu^\mathbb{P}\text{-a.e.} \quad (3.5)$$

*Example 3.11.* Theorem 3.89 in [20] states an Itô formula which transforms a special semimartingale  $X$  into a special semimartingale  $F(X_t)$  through a  $C^2$  function  $F : \mathbb{R} \rightarrow \mathbb{R}$ . There the process  $Y = X$  is supposed to fulfill Hypothesis 3.10

*Remark 3.12.* Let us suppose that  $\mu$  is the jump measure of a càdlàg process  $X$ . Hypothesis 3.10 holds for  $Y = X$ , with  $\tilde{\gamma}(t, \omega, x) = x$ .

The role of Hypothesis 3.10 is clarified by the following proposition.

**Proposition 3.13.** *Let  $Y$  be a càdlàg adapted process satisfying Hypothesis 3.10. Then, there exists a null set  $\mathcal{N}$  such that, for every Borel function  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  satisfying  $\varphi(s, 0) = 0$  for every  $s \in [0, T]$ , we have*

$$\sum_{s \leq T} \varphi(s, \Delta Y_s(\omega)) = \int_{[0, T] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, ds de), \quad \omega \notin \mathcal{N}. \quad (3.6)$$

*Proof.* Taking into account that  $\{\Delta Y \neq 0\} \subset D$  and the fact that  $\varphi(s, 0) = 0$ , it will be enough to prove that

$$\sum_{s \leq T} \varphi(s, \Delta Y_s(\omega)) \mathbb{1}_D(\omega, s) = \int_{[0, T] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, ds de), \quad \omega \notin \mathcal{N}, \quad (3.7)$$

for every Borel function  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$ .

Let  $(I_m)_m$  be a sequence of subsets of  $[0, T] \times \mathbb{R}$ , which is a  $\pi$ -system generating  $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$ . setting  $\varphi_m(s, x) = \mathbb{1}_{I_m}(s, x)$ , for every  $m$  we will show that

$$\sum_{s \leq T} \varphi_m(s, \Delta Y_s) \mathbb{1}_D(\cdot, s) = \int_{[0, T] \times \mathbb{R}} \varphi_m(s, \tilde{\gamma}(\cdot, s, e)) \mu(\cdot, ds de), \quad \text{a.s.} \quad (3.8)$$

As a matter of fact, let  $\phi : \Omega \rightarrow \mathbb{R}_+$  bounded,  $(\mathcal{F}_t)$ -measurable. Identity (3.8) holds if we show that the expectations of both sides against  $\phi$  are equal. We write

$$\begin{aligned} & \mathbb{E} \left[ \phi \int_{[0, T] \times \mathbb{R}} \varphi_m(s, \tilde{\gamma}(\cdot, s, e)) \mu(\cdot, ds de) \right] \\ &= \int_{\Omega \times ]0, T] \times \mathbb{R}} d\mathbb{P}(\omega) \mu(\omega, ds de) \phi(\omega) \varphi_m(s, \tilde{\gamma}(\omega, s, e)) \\ &= \int_{\Omega \times ]0, T] \times \mathbb{R}} dM_\mu^\mathbb{P}(\omega, s, e) \phi(\omega) \varphi_m(s, \tilde{\gamma}(\omega, s, e)) \\ &= \int_{\Omega \times ]0, T]} dM_\mu^\mathbb{P}(\omega, s, y) \phi(\omega) \varphi_m(s, \Delta Y_s(\omega)) \\ &= \int_{\Omega \times ]0, T] \times \mathbb{R}} d\mathbb{P}(\omega) \mu(\omega, ds de) \phi(\omega) \varphi_m(s, \Delta Y_s(\omega)) \\ &= \int_{\Omega} d\mathbb{P}(\omega) \phi(\omega) \sum_{0 < s \leq T} \mathbb{1}_D(\omega, s) \varphi_m(s, \Delta Y_s(\omega)) \int_{\mathbb{R}} \delta_{\beta_s(\omega)}(dx) \\ &= \mathbb{E} \left[ \phi \sum_{0 < s \leq T} \mathbb{1}_D(\cdot, s) \varphi_m(s, \Delta Y_s) \right], \end{aligned}$$

where we have used the form of  $\mu$  given by (B.3). Therefore, there exists a  $\mathcal{P}$ -null set  $\mathcal{N}_m$  such that

$$\sum_{0 < s \leq T} \varphi_m(s, \Delta Y_s(\omega)) \mathbb{1}_D(\omega, s) = \int_{[0, T] \times \mathbb{R}} \varphi_m(s, \tilde{\gamma}(\cdot, s, e)) \mu(\omega, ds de), \quad \omega \notin \mathcal{N}_m.$$

Define  $\mathcal{N} = \cup_m \mathcal{N}_m$ , then for  $\varphi = \varphi_m$  for every  $m$  we have

$$\sum_{0 < s \leq T} \varphi_m(s, \Delta Y_s(\omega)) \mathbb{1}_D(\omega, s) = \int_{[0, T] \times \mathbb{R}} \varphi_m(s, \tilde{\gamma}(\cdot, s, e)) \mu(\omega, ds de), \quad \omega \notin \mathcal{N}.$$

By a monotone class argument (see Theorem 21, Chapter 1, in [12]) the identity holds for every measurable bounded function  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , and therefore for every positive measurable function  $\varphi$  on  $[0, T] \times \mathbb{R}$  as well.  $\square$

We consider an additional assumption on a generic adapted process  $Z$ .

**Hypothesis 3.14.**  $Z$  is a càdlàg predictable process satisfying  $\{\Delta Z \neq 0\} \subset J$ .

We have the following result.

**Proposition 3.15.** Assume that  $X$  satisfy Hypotheses 3.2, with decomposition  $X = X^i + X^p$ , where  $X^i$  (resp.  $X^p$ ) fulfills Hypothesis 3.10 (resp. Hypothesis 3.14). Then, there exists a null set  $\mathcal{N}$  such that, for every Borel function  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  satisfying  $\varphi(s, 0) = 0$  for every  $s \in [0, T]$ , we have

$$\int_{[0, T] \times \mathbb{R}} \varphi(s, x) \mu^X(\omega, ds dx) = \int_{[0, T] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, ds de) + V^\varphi(\omega) \quad \text{for every } \omega \notin \mathcal{N}, \quad (3.9)$$

with  $V^\varphi(\omega) = \sum_{0 < s \leq T} \varphi(s, \Delta X_s^p(\omega))$ . In particular,

$$\int_{[0, T] \times \mathbb{R}} \varphi(s, x) \mu^X(\omega, ds dx) \geq \int_{[0, T] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, ds de) \quad \text{for every } \omega \notin \mathcal{N}. \quad (3.10)$$

Identity (3.9) still holds true when  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  and the left-hand side is finite.

*Proof.* Let  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$ . Taking into account Proposition 3.3-(i) and the fact that  $\varphi(s, 0) = 0$ , we have, for almost all  $\omega$ ,

$$\begin{aligned} & \sum_{0 < s \leq T} \varphi(s, \Delta X_s(\omega)) \\ &= \sum_{0 < s \leq T} \varphi(s, \Delta X_s^i(\omega) + \Delta X_s^p(\omega)) \mathbb{1}_{\{\Delta X^p=0\}}(\omega, s) + \sum_{s \leq T} \varphi(s, \Delta X_s^i(\omega) + \Delta X_s^p(\omega)) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s) \\ &= \sum_{0 < s \leq T} \varphi(s, \Delta X_s^i(\omega)) \mathbb{1}_{\{\Delta X^p=0\}}(\omega, s) + \sum_{s \leq T} \varphi(s, \Delta X_s^p(\omega)) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s) \\ &= \sum_{0 < s \leq T} \varphi(s, \Delta X_s^i(\omega)) + \sum_{s \leq T} \varphi(s, \Delta X_s^p(\omega)). \end{aligned}$$

By Proposition 3.13 applied to  $Y = X^i$ , there exists a null set  $\mathcal{N}$  such that, for every  $\omega \notin \mathcal{N}$ , previous expression gives

$$\int_{[0, T] \times \mathbb{R}} \varphi(s, x) \mu^X(\omega, ds dx) = \int_{[0, T] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, ds de) + \sum_{0 < s \leq T} \varphi(s, \Delta X_s^p(\omega)).$$

The second part of the statement holds decomposing  $\varphi = \varphi^+ - \varphi^-$ .  $\square$

*Remark 3.16.* The result in Proposition 3.15 still holds true if  $\varphi$  is a real-valued random function on  $\Omega \times [0, T] \times \mathbb{R}$ .

We will make the following assumption on  $\mu$ .

**Hypothesis 3.17.** (i)  $D = K \cup (\cup_n [[T_n^i]])$  up to an evanescent set, where  $(T_n^i)_n$  are totally inaccessible times such that  $[[T_n^i]] \cap [[T_m^i]] = \emptyset$ ,  $n \neq m$ ;

(ii) for every predictable time  $S$  such that  $[[S]] \subset K$ ,  $\nu(\{S\}, de) = \mu(\{S\}, de)$  a.s.

*Remark 3.18.* Hypothesis 3.17-(i) implies that  $J = K$ , up to an evanescent set, see Proposition B.13.

*Remark 3.19.* Let  $\nu$  denote the compensator of  $\mu$ .

(i)  $\nu$  admits a disintegration of the type

$$\nu(\omega, ds de) = dA_s(\omega) \phi(\omega, s, de), \quad (3.11)$$

where  $\phi$  is a random measure from  $(\Omega \times [0, T], \mathcal{P})$  into  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $A$  is a right-continuous nondecreasing predictable process, such that  $A_0 = 0$ , see (B.1).

(ii) Given  $\nu$  in the form (3.11), then the process  $A$  is continuous if and only if  $D = \cup_n [[T_n^i]]$ , where  $(T_n^i)_n$  are totally inaccessible times, see, e.g., Assumption (A) in [9]. In this case it follows that  $J = K = \emptyset$ , and consequently Hypothesis 3.17 trivially holds.

For instance  $A$  in (3.11) is continuous when  $\mu$  is a Poisson random measure, see, e.g., Chapter II, Section 4.b in [21].

We are ready to state the main result of the section.

**Proposition 3.20.** *Let  $\mu$  satisfy Hypothesis 3.17. Assume that  $X$  satisfy Hypothesis 3.2, with decomposition  $X = X^i + X^p$ , where  $X^i$  (resp.  $X^p$ ) fulfills Hypothesis 3.10 (resp. Hypothesis 3.14). Let  $\varphi : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  such that  $\varphi(\omega, s, 0) = 0$  for every  $s \in [0, T]$ , up to indistinguishability, and assume that there exists a  $\tilde{\mathcal{P}}$ -measurable subset  $A$  of  $\Omega \times [0, T] \times \mathbb{R}$  satisfying*

$$|\varphi| \mathbb{1}_A * \mu^X \in \mathcal{A}_{\text{loc}}^+, \quad |\varphi|^2 \mathbb{1}_{A^c} * \mu^X \in \mathcal{A}_{\text{loc}}^+. \quad (3.12)$$

Then

$$\int_{]0, t] \times \mathbb{R}} \varphi(s, x) (\mu^X - \nu^X)(ds dx) = \int_{]0, t] \times \mathbb{R}} \varphi(s, \tilde{\gamma}(s, e)) (\mu - \nu)(ds de) \quad \text{a.s.} \quad (3.13)$$

*Remark 3.21.* Under condition (3.12), Remark 2.6 and inequality (3.10) in Proposition 3.15 imply that  $\varphi(s, x) \in \mathcal{G}_{\text{loc}}^1(\mu^X)$  and  $\varphi(s, \tilde{\gamma}(s, e)) \in \mathcal{G}_{\text{loc}}^1(\mu)$ . In particular the two stochastic integrals in (3.13) are well-defined.

*Proof.* Clearly the result holds if we show that  $\varphi$  verifies (3.13) under one of the two following assumptions:

(i)  $|\varphi| * \mu^X \in \mathcal{A}_{\text{loc}}^+$ ,

(ii)  $|\varphi|^2 * \mu^X \in \mathcal{A}_{\text{loc}}^+$ .

By localization arguments, it is enough to show it when  $|\varphi| * \mu^X \in \mathcal{A}^+$ ,  $|\varphi|^2 * \mu^X \in \mathcal{A}^+$ . Below we will consider the first case, the second case will follow from the first one by approaching  $\varphi$  with  $\varphi(s, x) \mathbb{1}_{\varepsilon < |x| \leq 1/\varepsilon} \mathbb{1}_{s \in [0, T]}$  in  $\mathcal{L}^2(\mu^X)$ , and taking into account the fact that  $\mu^X$ , restricted to  $\varepsilon \leq |x| \leq 1/\varepsilon$ , is finite, since  $\mu^X$  is  $\sigma$ -finite.

Let us define

$$\begin{aligned} M_t &:= \int_{]0, t] \times \mathbb{R}} \varphi(\cdot, s, x) (\mu^X - \nu^X)(ds dx), \\ N_t &:= \int_{]0, t] \times \mathbb{R}} \varphi(\cdot, s, \tilde{\gamma}(\cdot, s, e)) (\mu - \nu)(ds de). \end{aligned} \quad (3.14)$$

Notice that the processes  $M$  and  $N$  are purely discontinuous local martingales, see e.g. Definition B.16. We have to prove that  $M$  and  $N$  are indistinguishable. To this end, by Corollary A.9, it is enough to prove that  $\Delta M = \Delta N$ , up to an evanescent set. Observe that

$$\begin{aligned}\Delta M_s &= \int_{\mathbb{R}} \varphi(\cdot, s, x) (\mu^X - \nu^X)(\{s\}, dx) \\ &= \int_{\mathbb{R}} \varphi(\cdot, s, x) (1 - \mathbb{1}_J(\cdot, s)) (\mu^X - \nu^X)(\{s\}, dx) + \int_{\mathbb{R}} \varphi(\cdot, s, x) \mathbb{1}_J(\cdot, s) (\mu^X - \nu^X)(\{s\}, dx),\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\Delta N_s &= \int_{\mathbb{R}} \varphi(\cdot, s, \tilde{\gamma}(\cdot, s, e)) (\mu - \nu)(\{s\}, de) \\ &= \int_{\mathbb{R}} \varphi(\cdot, s, \tilde{\gamma}(\cdot, s, e)) \mathbb{1}_J(\cdot, s) (\mu - \nu)(\{s\}, de) + \int_{\mathbb{R}} \varphi(\cdot, s, \tilde{\gamma}(\cdot, s, e)) (1 - \mathbb{1}_J(\cdot, s)) (\mu - \nu)(\{s\}, de).\end{aligned}\tag{3.16}$$

By definition of  $J$ , for every  $\omega$  and every  $s$  we have

$$\nu(\omega, \{s\}, de) (1 - \mathbb{1}_J(\omega, s)) = 0.\tag{3.17}$$

Moreover, since  $J$  is a predictable thin set, there exists a sequence of predictable times  $(R_n)_n$  with disjoint graphs, such that  $J = \cup_n [[R_n]]$ . We recall that Hypothesis 3.17-(i) implies that  $J = K$ , see Proposition B.13. By this fact, and taking into account Hypothesis 3.17-(ii), there exists a null set  $\mathcal{N}$ , such that, for every  $n \in \mathbb{N}$ ,  $\omega \notin \mathcal{N}$ ,

$$\mu(\omega, \{R_n(\omega)\}, de) \mathbb{1}_J(\omega, s) = \nu(\omega, \{R_n(\omega)\}, de) \mathbb{1}_J(\omega, s).$$

By additivity, it follows that for every  $\omega \notin \mathcal{N}$ , for every  $s \in [0, T]$ ,

$$\mu(\omega, \{s\}, de) \mathbb{1}_J(\omega, s) = \nu(\omega, \{s\}, de) \mathbb{1}_J(\omega, s).\tag{3.18}$$

On the other hand,  $\{\Delta X^p \neq 0\} \subset J$  by Hypothesis 3.14. Recalling that  $\{\Delta X^p \neq 0\} = \{(\omega, s) : \nu^X(\{s\} \times \mathbb{R}) > 0\}$  (see Proposition 3.4-1.), we have for almost every  $\omega$ , for every  $s \in [0, T]$ , we have

$$\nu^X(\omega, \{s\}, dx) \mathbb{1}_J(\omega, s) = \nu^X(\omega, \{s\}, dx) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s),\tag{3.19}$$

so that

$$\nu^X(\omega, \{s\}, dx) (1 - \mathbb{1}_J(\omega, s)) = \nu^X(\omega, \{s\}, dx) (1 - \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s)) = 0.\tag{3.20}$$

Now notice that there always exists a sequence of predictable times exhausting the jumps of  $X^p$ , see Remark 3.7. By means of Proposition 3.6 we can prove, similarly as we did in order to establish (3.18), that for every  $\omega \notin \mathcal{N}$ ,  $\mathcal{N}$  possibly enlarged, for every  $s \in [0, T]$ ,

$$\mu^X(\omega, \{s\}, dx) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s) = \nu^X(\omega, \{s\}, dx) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s).\tag{3.21}$$

Finally, we notice that  $\mu^X(\omega, \{s\}, dx) \mathbb{1}_J(\omega, s) = \mu^X(\omega, \{s\}, dx) \mathbb{1}_{J \cap \{\Delta X \neq 0\}}(\omega, s)$ . Taking into account that  $X^i$  is a càdlàg quasi-left continuous process, by Definition A.18 we have

$$\begin{aligned}J \cap \{\Delta X \neq 0\} &= (\cup_n [[R_n]] \cap \{\Delta X^i \neq 0\}) \cup (\cup_n [[R_n]] \cap \{\Delta X^p \neq 0\}) \\ &= \cup_n [[R_n]] \cap \{\Delta X^p \neq 0\} = \{\Delta X^p \neq 0\}.\end{aligned}$$

This implies for every  $\omega \notin \mathcal{N}$ , and for every  $s \in [0, T]$ ,

$$\mu^X(\omega, \{s\}, dx) \mathbb{1}_J(\omega, s) = \mu^X(\omega, \{s\}, dx) \mathbb{1}_{J \cap \{\Delta X \neq 0\}}(\omega, s)$$

$$= \mu^X(\omega, \{s\}, dx) \mathbb{1}_{\{\Delta X^p \neq 0\}}(\omega, s). \quad (3.22)$$

Collecting (3.19), (3.21) and (3.22) we conclude that for every  $\omega \notin \mathcal{N}$ , for every  $s \in [0, T]$ , for every  $n \in \mathbb{N}$ ,

$$\mu^X(\omega, \{s\}, dx) \mathbb{1}_J(\omega, s) = \nu^X(\omega, \{s\}, dx) \mathbb{1}_J(\omega, s). \quad (3.23)$$

Therefore, for every  $\omega \notin \mathcal{N}$ , for every  $s \in [0, T]$ , taking into account (3.17), (3.18), (3.20), (3.23), expressions (3.15) and (3.16) become

$$\Delta M_s = \int_{\mathbb{R}} \varphi(s, x) (1 - \mathbb{1}_J(s)) \mu^X(\{s\}, dx), \quad (3.24)$$

$$\Delta N_s = \int_{\mathbb{R}} \varphi(s, \tilde{\gamma}(s, e)) (1 - \mathbb{1}_J(s)) \mu(\{s\}, de). \quad (3.25)$$

Now let us prove that, for every  $s \in [0, T]$ ,  $\Delta M_s(\omega) = \Delta N_s(\omega)$  for every  $\omega \notin \mathcal{N}$ , namely up to an evanescent set. Set

$$\varphi_s(\omega, t, x) := \varphi(\omega, t, x) (1 - \mathbb{1}_J(\omega, t)) \mathbb{1}_{\{s\}}(t),$$

then  $\Delta M_s$  and  $\Delta N_s$  can be rewritten as

$$\begin{aligned} \Delta M_s(\omega) &= \int_{[0, T] \times \mathbb{R}} \varphi_s(\omega, t, x) \mu^X(\omega, dt dx), \\ \Delta N_s(\omega) &= \int_{[0, T] \times \mathbb{R}} \varphi_s(\omega, t, \tilde{\gamma}(\omega, t, e)) \mu(\omega, dt de), \end{aligned}$$

Then, Proposition 3.15 applied to the process  $\varphi_s$  implies that (possibly enlarging the null set  $\mathcal{N}$ ),

$$\int_{]0, T] \times \mathbb{R}} \varphi_s(\omega, t, x) \mu^X(\omega, dt dx) = \int_{]0, T] \times \mathbb{R}} \varphi_s(t, \tilde{\gamma}(\omega, t, e)) \mu(\omega, dt de) + V^{\tilde{\varphi}}(\omega) \quad \text{for every } \omega \notin \mathcal{N},$$

or, equivalently, that

$$\int_{\mathbb{R}} \varphi(\omega, s, x) \mu^X(\omega, \{s\}, dx) = \int_{\mathbb{R}} \varphi(\omega, s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, \{s\}, de) + V^{\tilde{\varphi}}(\omega), \quad \text{for every } \omega \notin \mathcal{N},$$

where

$$V^{\varphi_s}(\omega) = \sum_{t \leq T} \varphi_s(\omega, t, \Delta X_t^p(\omega)) = \varphi(\omega, s, \Delta X_s^p(\omega)) \mathbb{1}_{J^c \cap \{\Delta X^p \neq 0\}}(\omega, s). \quad (3.26)$$

Recalling that  $\{\Delta X^p \neq 0\} \subset J$  by Hypothesis 3.14, it straightly follows from (3.26) that  $V^{\varphi_s}(\omega)$  is zero. In particular, up to an evanescent set, we have

$$\int_{\mathbb{R}} \varphi(\omega, s, x) \mu^X(\omega, \{s\}, dx) = \int_{\mathbb{R}} \varphi(s, \tilde{\gamma}(\omega, s, e)) \mu(\omega, \{s\}, de),$$

in other words  $\Delta M = \Delta N$  up to an evanescent set, and this concludes the proof.  $\square$

We end the section focusing on the case when  $X$  is of jump-diffusion type.

**Lemma 3.22.** *Let  $\mu$  satisfy Hypothesis 3.17. Let  $N$  be a continuous martingale, and  $B$  an increasing predictable càdlàg process, with  $B_0 = 0$ , such that  $\{\Delta B \neq 0\} \subset J$ . Let  $X$  be a process which is solution of equation*

$$X_t = X_0 + \int_0^t b(s, X_{s-}) dB_s + \int_0^t \sigma(s, X_s) dN_s + \int_{]0, t] \times \mathbb{R}} \gamma(s, X_{s-}, e) (\mu - \nu)(ds de), \quad (3.27)$$

for some given Borel functions  $b, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t |b(s, X_{s-})| dB_s < \infty \text{ a.s.}, \quad (3.28)$$

$$\int_0^t |\sigma(s, X_s)|^2 d[N, N]_s < \infty \text{ a.s.}, \quad (3.29)$$

$$(\omega, s, e) \mapsto \gamma(s, X_{s-}(\omega), e) \in \mathcal{G}_{\text{loc}}^1(\mu). \quad (3.30)$$

Then  $X$  satisfies Hypothesis 3.2, with decomposition  $X = X^i + X^p$ , where

$$X_t^i = \int_{]0, t] \times \mathbb{R}} \gamma(s, X_{s-}, e) (\mu - \nu)(ds de), \quad (3.31)$$

$$X_t^p = X_0 + \int_0^t b(s, X_{s-}) dB_s + \int_0^t \sigma(s, X_s) dN_s. \quad (3.32)$$

Moreover, the process  $X^i$  fulfills Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = \gamma(s, X_{s-}(\omega), e) (1 - \mathbb{1}_K(\omega, s))$ , and the process  $X^p$  satisfies Hypothesis 3.14.

*Proof.* Since  $N$  is continuous, it straight follows from (3.32) that

$$\Delta X_s^p = b(s, X_{s-}) \Delta B_s. \quad (3.33)$$

We remark that  $X^i$  in (3.31) has the same expression as  $N$  defined in (3.14) where the integrand  $\varphi(\omega, s, \tilde{\gamma}(\omega, s, e))$  is replaced by  $\gamma(s, X_{s-}(\omega), e)$ . We recall that Hypothesis 3.17-(i) implies that  $J = K$ , see Proposition B.13. Similarly as for (3.25), we get

$$\Delta X_s^i = \int_{\mathbb{R}} \gamma(s, X_{s-}, e) (1 - \mathbb{1}_K(s)) \mu(\{s\}, de), \quad (3.34)$$

Since by Hypothesis 3.17  $D \setminus K = \cup_n [[T_n^i]]$  ( $(T_n^i)_n$  being a sequence of totally inaccessible times with disjoint graphs), (3.34) can be rewritten as

$$\Delta X_s^i(\omega) = \gamma(s, X_{s-}(\omega), \beta_s(\omega)) \mathbb{1}_{\cup_n [[T_n^i]]}(\omega, s). \quad (3.35)$$

We can easily show that the process  $X$  satisfies Hypothesis 3.2, namely  $X^p$  and  $X^i$  are respectively a càdlàg predictable process and a càdlàg quasi-left continuous adapted process. The fact that  $X^p$  is predictable straight follow from (3.32). Concerning  $X^i$ , let  $S$  be a predictable time; it is enough to prove that  $\Delta X_S^i \mathbb{1}_{\{S < \infty\}} = 0$  a.s., see Definition A.18. Identity (3.35) gives

$$\Delta X_S^i(\omega) \mathbb{1}_{\{S < \infty\}} = \gamma(S, X_{S-}(\omega), \beta_S(\omega)) \mathbb{1}_{\cup_n [[T_n^i]]}(\omega, S(\omega)) \mathbb{1}_{\{S < \infty\}}. \quad (3.36)$$

Since the graphs of the totally inaccessible times  $T_n^i$  are disjoint,  $\mathbb{1}_{\cup_n [[T_n^i]]}(\omega, S(\omega)) \mathbb{1}_{\{S < \infty\}} = \sum_n \mathbb{1}_{[[T_n^i]]}(\omega, S(\omega)) \mathbb{1}_{\{S < \infty\}}$ , and the conclusion follows by the definition of a totally inaccessible time, taking into account that  $S$  is a predictable time, see Remark A.15.

The process  $X^p$  in (3.32) satisfies Hypothesis 3.14. Indeed, by (3.33) we have

$$\{\Delta X^p \neq 0\} \subset \{\Delta B \neq 0\} \subset J = K. \quad (3.37)$$

Finally, we show that the process  $X^i$  in (3.31) fulfills Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = \gamma(s, X_{s-}(\omega), e) (1 - \mathbb{1}_K(\omega, s))$ . First, the fact that  $\{\Delta X^i \neq 0\} \subset D$  directly follows from (3.34). To prove  $\Delta X_s^i(\omega) = \tilde{\gamma}(\omega, s, \cdot)$ ,  $dM_\mu^{\mathbb{P}}(\omega, s)$ -a.e. it is enough to show that

$$\mathbb{E} \left[ \int_{]0, T] \times \mathbb{R}} \mu(\omega, ds de) |\tilde{\gamma}(\omega, s, e) - \Delta X_s^i(\omega)| \right] = 0.$$



To establish this, by the structure of  $\mu$ ,

$$\mathbb{E} \left[ \int_{]0, T] \times \mathbb{R}} \mu(\omega, ds de) |\tilde{\gamma}(\omega, s, e) - \Delta X_s^i(\omega)| \right] = \sum_{s \in ]0, T]} \mathbb{E} [\mathbb{1}_D(\cdot, s) |\tilde{\gamma}(\cdot, s, \beta_s(\cdot)) - \Delta X_s^i(\cdot)|]$$

which vanishes taking into account (3.35).  $\square$

## 4 Application to BSDEs

### 4.1 About BSDEs driven by an integer-valued random measure

Let  $\mu$  be an integer-valued random measure defined on  $\Omega \times \mathcal{B}([0, T] \times \mathbb{R})$ . Let  $M$  be a continuous process with  $M_0 = 0$ . Let  $(\mathcal{F}_t)$  be the canonical filtration associated to  $\mu$  and  $M$ , and suppose that  $M$  is an  $(\mathcal{F}_t)$ -local martingale. Let  $\tilde{g} : \Omega \times [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\tilde{f} : \Omega \times [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  be two measurable functions. The domain of  $\tilde{f}$  (resp.  $\tilde{g}$ ) is equipped with the  $\sigma$ -field  $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R}^3)$  (resp.  $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R}^2)$ ). Let  $\lambda$  be a predictable random measure on  $\Omega \times \mathcal{B}([0, T] \times \mathbb{R})$ . Let  $\zeta$  be a non-decreasing adapted continuous process, and  $\xi$  a square integrable random variable.  $\nu$  will denote a "good" version of the dual predictable projection of  $\mu$  in the sense of Proposition B.11. In particular,  $\nu(\omega, \{t\} \times \mathbb{R}) \leq 1$  identically.

We consider now the general BSDE

$$\begin{aligned} Y_t = & \xi + \int_{]t, T]} \tilde{g}(s, Y_{s-}, Z_s) d\zeta_s + \int_{]t, T] \times \mathbb{R}} \tilde{f}(s, e, Y_{s-}, U_s(e)) \lambda(ds de) \\ & - \int_{]t, T]} Z_s dM_s - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de) \end{aligned} \quad (4.1)$$

which constitutes equation (1.1) of the Introduction.

*Remark 4.1.* A general BSDE of type (4.1) is considered for instance in [33] (see formula (1.1)), with the following restrictions on the random measures  $\lambda$  and  $\nu$ :

$$\begin{aligned} \lambda([0, T] \times \mathbb{R}) & \text{ is a bounded random variable, } \lambda([0, t] \times \mathbb{R}) \text{ is continuous with respect to } t, \\ \nu([0, t] \times \mathbb{R}) & \text{ is continuous with respect to } t. \end{aligned} \quad (4.2)$$

The author proves (see Theorem 3.2. in [33]) that under suitable assumptions on the coefficients  $(\xi, \tilde{f}, \tilde{g})$  there exists a unique triplet of processes  $(Y, Z, U) \in \mathcal{L}^2(\zeta\lambda) \times \mathcal{L}^2(M) \times \mathcal{L}^2(\mu)$ , with  $\mathbb{E}[\sup_{t \in [0, T]} Y_t^2] < \infty$ , satisfying BSDE (1.1), where

$$\begin{aligned} \mathcal{L}^2(\zeta\lambda) & := \left\{ \text{optional processes } (Y_t)_{t \in [0, T]} : \mathbb{E} \left[ \int_0^T Y_s^2 d\zeta_s \right] + \mathbb{E} \left[ \int_0^T Y_s^2 \lambda(ds, \mathbb{R}) \right] < \infty \right\}, \\ \mathcal{L}^2(M) & := \left\{ \text{predictable processes } (Z_t)_{t \in [0, T]} : \mathbb{E} \left[ \int_0^T Z_s^2 d\langle M \rangle_s \right] < \infty \right\}, \end{aligned}$$

and  $\mathcal{L}^2(\mu)$  is the space introduced in (B.20).

In the sequel we will consider stochastic processes related to the random measure  $\mu$  in the following way.

**Hypothesis 4.2.** *X is an adapted càdlàg process verifying Hypothesis 3.2 with decomposition  $X = X^i + X^p$ , where  $X^i$  (resp.  $X^p$ ) fulfills Hypothesis 3.10 with some predictable process  $\tilde{\gamma}$  (resp. fulfills Hypothesis 3.14), with respect to the random measure  $\mu$ .*

We consider some important examples.

*Example 4.3.* Let us focus on the BSDE

$$Y_t = g(X_T) + \int_{[t, T]} f(s, X_s, Y_s, Z_s, U_s(\cdot)) ds - \int_{[t, T]} Z_s dW_s - \int_{[t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de), \quad (4.3)$$

which constitutes a particular case of the BSDE (4.1). This is considered for instance in [3]. Here  $W$  is a Brownian motion and  $\mu(ds de)$  is a Poisson random measure with compensator

$$\nu(ds de) = \lambda(de) ds, \quad (4.4)$$

where  $\lambda$  is a Borel  $\sigma$ -finite measure on  $\mathbb{R} \setminus \{0\}$  and

$$\int_{\mathbb{R}} (1 \wedge |e|^2) \lambda(de) < +\infty. \quad (4.5)$$

Poisson random measures have been introduced for instance in Chapter II, Section 4.b in [21]. The process  $X$  appearing in (4.3) is a Markov process satisfying the SDE

$$dX_s = b(X_s) ds + \sigma(X_s) dW_s + \int_{\mathbb{R}} \gamma(X_{s-}, e) (\mu - \nu)(ds de), \quad s \in [t, T], \quad (4.6)$$

where  $b : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  are globally Lipschitz, and  $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function such that, for some real  $K$ , and for all  $e \in \mathbb{R}$ ,

$$\begin{cases} |\gamma(x, e)| \leq K (1 \wedge |e|), & x \in \mathbb{R}, \\ |\gamma(x_1, e) - \gamma(x_2, e)| \leq K |x_1 - x_2| (1 \wedge |e|) & x_1, x_2 \in \mathbb{R}. \end{cases} \quad (4.7)$$

For every starting point  $x \in \mathbb{R}$  and initial time  $t \in [0, T]$ , there is a unique solution to (4.6) denoted  $X^{t,x}$  (see [3], Section 1). Moreover, modulo suitable assumptions on the coefficients  $(g, f)$ , it is proved that the BSDE (4.3) admits a unique solution  $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{L}^2(\mu)$ , see Theorem 2.1 in [3], where

$$\begin{aligned} \mathcal{S}^2 &:= \left\{ \text{adapted càdlàg processes } (Y_t)_{t \in [0, T]} : \left\| \sup_{t \in [0, T]} |Y_t| \right\|_{L^2(\Omega)} < \infty \right\}, \\ \mathcal{L}^2 &:= \left\{ \text{predictable processes } (Z_t)_{t \in [0, T]} : \mathbb{E} \left[ \int_0^T Z_s^2 ds \right] < \infty \right\}, \\ \mathcal{L}^2(\mu) &:= \left\{ \text{predictable random fields } (U_s(\cdot))_{s \in [0, T]} : \mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}} |U_s(e)|^2 \nu(ds de) \right] < \infty \right\}. \end{aligned}$$

When  $X = X^{t,x}$  the solution  $(Y, Z)$  of (4.3) is denoted  $(Y^{t,x}, Z^{t,x})$ . In [3] it is proved that

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad (4.8)$$

satisfies  $Y_s^{t,x} = u(s, X_s^{t,x})$  for every  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $s \in [t, T]$ .  $\square$

**Lemma 4.4.** *Let  $\mu$  and  $X$  be respectively the Poisson random measure and the stochastic process satisfying the SDE (4.6) in Example 4.3. Then  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with respect to  $\mu$ , with decomposition  $X = X^i + X^p$ ,*

$$X_t^i = \int_{[0, t] \times \mathbb{R}} \gamma(X_{s-}, e) (\mu - \nu)(ds de), \quad (4.9)$$

$$X_t^p = \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s. \quad (4.10)$$

In particular  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = \gamma(X_{s-}(\omega), e)$ .

*Proof.* Our aim is to apply Lemma 3.22. We start by noticing that  $\nu$  in (4.4) is in the form (3.11) with  $A_s = s$ . Therefore Hypothesis 3.17 is verified, see Remark 3.19-(ii). On the other hand, the process  $X$  satisfies the stochastic differential equation (4.6), which is a particular case of (3.27) when  $B_s = s$ ,  $N_s = W_s$ , and  $b, \sigma, \gamma$  are homogeneous.  $b$  and  $\sigma$  verify (3.28), (3.29) since they have linear growth. Condition (3.30) can be verified using the characterization of  $\mathcal{G}_{\text{loc}}^1(\mu)$  in Theorem B.19. In that context, setting  $W(\omega, s, e) = \gamma(s, X_{s-}(\omega), e)$ , we have  $\hat{W} = 0$ , and we have to verify that  $|W|^2 \mathbb{1}_{\{|W| \leq 1\}} * \nu + |W| \mathbb{1}_{\{|W| > 1\}} * \nu$  belongs to  $\mathcal{A}_{\text{loc}}^+$ . This follows from (4.5) and (4.7).

Then, by Lemma 3.22,  $X$  verifies Hypothesis 3.2, with decomposition  $X = X^i + X^p$ , where  $X^i$  and  $X^p$  are given respectively by (4.9) and (4.10). Moreover, the process  $X^i$  fulfills Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = \gamma(X_{s-}(\omega), e)$ , and the process  $X^p$  satisfies Hypothesis 3.14.  $\square$

When  $\zeta$  and  $M$  vanish, BSDE (4.1) turns out to be driven only by a purely discontinuous martingale, and becomes

$$Y_t = \xi + \int_{]t, T]} \tilde{f}(s, \omega, e, Y_{s-}, U_s(e)) \lambda(ds de) - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de). \quad (4.11)$$

Below we consider two significant cases, given respectively in Examples 4.5 and 4.7.

*Example 4.5.* In [8] the authors study a BSDE driven by an integer-valued random measure  $\mu$  associated to a given pure jump Markov process  $X$ , of the form

$$Y_t = g(X_T) + \int_{]t, T]} f(s, X_s, Y_s, U_s(\cdot)) ds - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de). \quad (4.12)$$

The underlying process  $X$  is generated by a marked point process  $(T_n, \zeta_n)$ , where  $(T_n)_n$  are increasing random times such that  $T_n \in ]0, \infty[$ , where either the times  $(T_n)_n$  are a finite number or  $\lim_{n \rightarrow \infty} T_n = +\infty$ , and  $\zeta_n$  are random variables in  $\mathbb{R}$ , see e.g. Chapter III, Section 2 b., in [20]. This means that  $X$  is a càdlàg process such that  $X_t = \zeta_n$  for  $t \in [T_n, T_{n+1}[$ , for every  $n \in \mathbb{N}$ . In particular,  $X$  has a finite number of jumps on each compact. The associated integer-valued random measure  $\mu$  is the sum of the Dirac measures concentrated at the marked point process  $(T_n, \zeta_n)$ , and can be written as

$$\mu(ds de) = \sum_{s \in [0, T]} \mathbb{1}_{\{X_{s-} \neq X_s\}} \delta_{(s, X_s)}(dt de). \quad (4.13)$$

Given a measure  $\mu$  in the form (4.13), it is related to the jump measure  $\mu^X$  in the following way: for every Borel subset  $A$  of  $\mathbb{R}$ ,

$$\int_{]0, T] \times \mathbb{R}} \mathbb{1}_A(e - X_{s-}) \mu(ds de) = \int_{]0, T] \times \mathbb{R}} \mathbb{1}_A(x) \mu^X(ds dx). \quad (4.14)$$

This is for instance explained in Example 3.22 in [20]. The pure jump process  $X$  then satisfies the equation

$$X_t = X_0 + \sum_{s \leq t} \Delta X_s = X_0 + \int_{]0, t] \times \mathbb{R}} (e - X_{s-}) \mu(ds de). \quad (4.15)$$

The compensator of  $\mu(ds de)$  is

$$\nu(ds de) = \lambda(s, X_{s-}, de) ds, \quad (4.16)$$

where  $\lambda$  is the transition rate measure of the process satisfying

$$\sup_{t \in [0, T], x \in \mathbb{R}} \lambda(t, x, \mathbb{R}) < \infty, \quad (4.17)$$

see Section 2.1 in [8].

Under suitable assumptions on the coefficients  $(g, f)$ , Theorem 3.4 in [8] states that the BSDE (4.12) admits a unique solution  $(Y, U) \in \mathcal{L}^2 \times \mathcal{L}^2(\mu)$ , where  $\mathcal{L}^2(\mu)$  and  $\mathcal{L}^2$  are the spaces introduced in Example 4.3.

Theorem 4.4 in [8] shows moreover that there exists a measurable function  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\forall e \in E, t \mapsto u(t, e) \text{ is absolutely continuous on } [0, T], \quad (4.18)$$

$$u(s, X_s) \in \mathcal{L}^2 \text{ and } u(s, e) - u(s, X_{s-}) \in \mathcal{L}^2(\mu), \quad s \in [0, T], \quad (4.19)$$

and the unique solution of the BSDE (4.12) can be represented as

$$Y_s = u(s, X_s), \quad s \in [0, T], \quad (4.20)$$

$$U_s(e) = u(s, e) - u(s, X_{s-}), \quad \lambda(s, X_{s-}, de) ds\text{-a.e. } s \in [0, T]. \quad (4.21)$$

□

**Lemma 4.6.** *Let  $X$  and  $\mu$  be respectively a pure jump Markov process and the corresponding integer-valued random measure as in Example 4.5. Then  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with decomposition  $X = X^i$ ,  $X^p = 0$ . In particular,  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = e - X_{s-}(\omega)$ .*

*Proof.* Since  $\nu$  in (4.16) is in the form (3.11) with  $A_s = s$ , Hypothesis 3.17 is verified, see Remark 3.19-(ii).

The process  $X^i = X$  satisfies (4.15). Recalling the relation (4.14) between  $\mu$  and  $\mu^X$ , the continuity of the above mentioned process  $A$  also implies that  $X = X^i$  is quasi-left continuous, see Corollary B.9. Finally, by definition of  $\mu$  we have

$$\mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}} \mu(ds de) |(e - X_{s-}) - \Delta X_s| \right] = 0,$$

therefore  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = e - X_{s-}(\omega)$ .

□

We start now describing the second example. In the recent paper [1], one studies the existence and uniqueness for a BSDE driven by a purely discontinuous martingale of the form

$$Y_t = \xi + \int_{]t, T]} \tilde{f}(s, Y_{s-}, U_s(\cdot)) dA_s - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de). \quad (4.22)$$

Here  $\mu(ds de)$  is an integer-valued random measure with compensator  $\nu(ds de) = dA_s \phi_s(de)$ , where  $\phi$  is a probability kernel and  $A$  is a right-continuous nondecreasing predictable process, such that  $\hat{\nu}_s(\mathbb{R}) = \Delta A_s \leq 1$  for every  $s$ . For any positive constant  $\beta$ ,  $\mathcal{E}^\beta$  will denote the Doléans-Dade exponential of the process  $\beta A$ . We consider the weighted spaces

$$\mathcal{L}_\beta^2(A) := \left\{ \text{adapted càdlàg processes } (Y_s)_{s \in [0, T]}, \text{ s.t. } \mathbb{E} \left[ \int_0^T \mathcal{E}_s^\beta |Y_{s-}|^2 dA_s \right] < \infty \right\},$$

$\mathcal{G}_\beta^2(\mu) := \left\{ \text{predictable processes } (U_s(\cdot))_{s \in [0, T]}, \text{ s.t.} \right.$

$$\left. \|U\|_{\mathcal{G}_\beta^2(\mu)}^2 := \mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}} \mathcal{E}_s^\beta |U_s(e) - \hat{U}_s|^2 \nu(ds de) + \sum_{s \in [0, T]} \mathcal{E}_s^\beta |\hat{U}_s|^2 (1 - \Delta A_s) \right] < \infty \right\}.$$

A solution to equation (4.22) with data  $(\beta, \xi, \tilde{f})$  is a pair  $(Y, Z) \in \mathcal{L}_\beta^2(A) \times \mathcal{G}_\beta^2(\mu)$  satisfying equation (4.22). We say that equation (4.22) admits a unique solution in  $\mathcal{L}_\beta^2(A) \times \mathcal{G}_\beta^2(\mu)$  if, given two solutions  $(Y, U)$ ,  $(Y', U')$ , we have  $Y_t = Y'_t$   $d\mathbb{P} \otimes dA_t$ -a.e. and  $\|U - U'\|_{\mathcal{G}_\beta^2(\mu)}^2 = 0$  (in particular  $\|U - U'\|_{\mathcal{G}^2(\mu)}^2 = 0$ ).

In [1] one requires suitable assumptions on the triplet  $(\tilde{f}, \xi, \beta)$ . In particular  $\tilde{f}$  is of Lipschitz type in the third and fourth variable and  $\xi$  is a square integrable random variable with some weight. Moreover, the following technical assumption has to be fulfilled: there exists  $\varepsilon \in ]0, 1[$  such that

$$2|L_y|^2 |\Delta A_t|^2 \leq 1 - \varepsilon, \quad \mathbb{P}\text{-a.s.}, \quad \forall t \in [0, T], \quad (4.23)$$

where  $L_y$  is the Lipschitz constant of  $\tilde{f}$  with respect to  $y$ . Under these hypotheses, for  $\beta$  large enough, it can be proved that there exists a unique solution  $(Y, U) \in \mathcal{L}_\beta^2(A) \times \mathcal{G}_\beta^2(\mu)$  to BSDE (4.22), see Theorem 4.1 in [1].

At this point some comments may be useful. Two random fields  $U$  and  $U'$  in  $\mathcal{G}_{\text{loc}}^2(\mu)$  will be said to be equal if  $U = U'$   $\mathbb{M}_\nu^\mathbb{P}$ -a. e. (i.e.,  $d\mathbb{P}(\omega) \nu(\omega, dt de)$ -a.e.).

Uniqueness in Theorem 4.1 in [1] means the following: if  $(Y, U)$ ,  $(Y', U')$  are solutions of the BSDE (4.22), then  $Y = Y'$  and, by Proposition B.28, there is a predictable process  $(l_s)$  such that  $U(\cdot) - U'(\cdot) = l \mathbb{1}_K$ ,  $\nu$ -a.e.

Given a solution  $(Y, U_0)$  of BSDE (4.22), the class of all solutions will be given by the pairs  $(Y, U)$ , where  $U = l \mathbb{1}_K + U_0$  for some predictable process  $(l_s)$ . In particular, if  $K = \emptyset$ , then the second component of the BSDE solution is unique in the smaller space  $\mathcal{L}^2(\mu)$ .

*Example 4.7.* Let us now consider a particular case of BSDE (4.22), namely a BSDE driven by the integer-valued random measure  $\mu$  associated to a given Markov process  $X$ , of the form

$$Y_t = g(X_T) + \int_{[t, T]} f(s, X_{s-}, Y_{s-}, U_s(\cdot)) dA_s - \int_{[t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de), \quad (4.24)$$

where  $\mu(ds de)$  is an integer-valued random measure with compensator  $\nu(ds de) = dA_s \phi_s(de)$ , where  $(\phi_s)_{s \in [0, T]}$  is a random probability kernel and  $A$  is a right-continuous nondecreasing predictable process.

We assume that  $X$  is a Piecewise Deterministic Markov Process (PDMP) associated to the random measure  $\mu$ , with values in the interval  $]0, 1[$ . Such a process has random jumps  $(T_n)_n$  and a deterministic motion between jumps according to a drift  $h : ]0, 1[ \rightarrow \mathbb{R}$  which is Lipschitz continuous. When the process reaches the boundary, it will instantaneously jump inside the interval. We will follow the notations in [10], Chapter 2, Section 24 and 26. For every  $x \in ]0, 1[$ , we will express by  $t_*(x)$  the first time such that the process  $X$  starting at  $x$  reaches 0 or 1. The behavior of  $X$  is described by a triplet of local characteristics  $(h, \lambda, Q)$ , where  $h$  is the drift introduced before,  $\lambda : ]0, 1[ \rightarrow \mathbb{R}$  is a measurable function satisfying

$$\sup_{x \in ]0, 1[} |\lambda(x)| < \infty, \quad (4.25)$$

and  $Q$  is a probability transition measure on  $[0, 1] \times \mathcal{B}(]0, 1[)$ , such that

$$\text{for some } \varepsilon > 0, Q(x, B_\varepsilon) = 1 \text{ for } x \in \{0, 1\}, \text{ where } B_\varepsilon = \{x \in ]0, 1[: t_*(x) > \varepsilon\}. \quad (4.26)$$

Set  $N_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{t \geq T_n}$ . By Proposition 24.6 in [10], under conditions (4.25) and (4.26) we have

$$\mathbb{E}[N_t] < \infty \quad \forall t \in \mathbb{R}_+. \quad (4.27)$$

Notice that the PDMP  $X$  verifies the equation

$$X_t = X_0 + \int_0^t h(X_s) ds + \sum_{s \leq t} \Delta X_s. \quad (4.28)$$

In particular  $X$  admits a finite number of jumps on each compact interval. By 26.9 in [10], the random measure  $\mu$  is

$$\mu(ds de) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{X_{T_n} \in ]0, 1[ \}} \delta_{(T_n, X_{T_n})}(ds de) = \sum_{s \in [0, T]} \mathbb{1}_{\{X_{s-} \neq X_s\}} \delta_{(s, X_s)}(ds de), \quad (4.29)$$

which is of the type of (4.13). This implies the validity of (4.14), so that (4.28) can be rewritten as

$$X_t = X_0 + \int_0^t h(X_s) ds + \int_{]0, t] \times ]0, 1[} (e - X_{s-}) \mu(ds de).$$

In the following, by abuse of notations,  $\mu$  will denote the trivial extension of previous measure to the real line. In particular (4.28) can be reexpressed as

$$X_t = X_0 + \int_0^t h(X_s) ds + \int_{]0, t] \times \mathbb{R}} (e - X_{s-}) \mu(ds de). \quad (4.30)$$

The knowledge of  $(h, \lambda, Q)$  completely specifies the dynamics of  $X$ , see section 24 in [10]. According to (26.2) in [10], the compensator of  $\mu$  has the form

$$\nu(ds de) = (\lambda(X_{s-}) ds + dp_s^*) Q(X_{s-}, de), \quad (4.31)$$

where

$$p_t^* = \sum_{n=1}^{\infty} \mathbb{1}_{\{t \geq T_n\}} \mathbb{1}_{\{X_{T_n-} \in \{0, 1\}\}} \quad (4.32)$$

is the process counting the number of jumps of  $X$  from the boundary of its domain.

From (4.31) we can choose  $A_s$  and  $\phi_s(de)$  such that  $dA_s = \lambda(X_{s-}) ds + dp_s^*$  and  $\phi_s(de) = Q(X_{s-}, de)$ . In particular,  $A$  is predictable (not deterministic) and discontinuous, with jumps

$$\Delta A_s(\omega) = \hat{\nu}_s(\omega, \mathbb{R}) = \Delta p_s^*(\omega) = \mathbb{1}_{\{X_{s-}(\omega) \in \{0, 1\}\}}. \quad (4.33)$$

Consequently,  $\hat{\nu}_t(\omega, \mathbb{R}) > 0$  if and only if  $\hat{\nu}_t(\omega, \mathbb{R}) = 1$ , so that

$$J = \{(\omega, t) : \hat{\nu}_t(\omega, \mathbb{R}) > 0\} = \{(\omega, t) : \hat{\nu}_t(\omega, \mathbb{R}) = 1\} = K, \quad (4.34)$$

and

$$K = \{(\omega, t) : X_{t-}(\omega) \in \{0, 1\}\}. \quad (4.35)$$

□

**Lemma 4.8.** *Let  $X$  be the PDMP process we have considered in Example 4.7. Then*

$$\int_{]0, \cdot] \times \mathbb{R}} |e - X_{s-}| \nu(ds de) \in \mathcal{A}_{\text{loc}}^+.$$

*Proof.* We start by noticing that

$$\int_{]0, T] \times \mathbb{R}} |e - X_{s-}| \nu(ds de) < \infty \quad \text{a.s.}$$

Indeed

$$\begin{aligned} \int_{]0, T] \times \mathbb{R}} |e - X_{s-}| \nu(ds de) &= \int_{]0, T] \times ]0, 1[} |e - X_{s-}| (\lambda(X_{s-}) ds + dp_s^*) Q(X_{s-}, de) \\ &\leq \|\lambda\|_\infty (T + p_T^*). \end{aligned}$$

For every  $t \in [0, T]$  the jumps of the process

$$Y_t := \int_{]0, t] \times \mathbb{R}} |e - X_{s-}| \nu(ds de)$$

are given by

$$\Delta Y_t := \int_{]0, 1[} |e - X_{t-}| \hat{\nu}_t(de) \leq \hat{\nu}_t(\mathbb{R}) \leq 1.$$

Since  $Y_t$  has bounded jumps, it is a locally bounded process and therefore it belongs to  $\mathcal{A}_{\text{loc}}^+$ , see for instance the proof of Corollary in pag 373 in [29].  $\square$

**Lemma 4.9.** *Let  $\mu$  and  $X$  be respectively the random measure and the associated PDMP satisfying equation (4.30) in Example 4.7. Assume in addition that there exists a function  $\beta : \{0, 1\} \rightarrow ]0, 1[$ , such that*

$$X_s = \beta(X_{s-}) \quad \text{on } \{(\omega, s) : X_{s-}(\omega) \in \{0, 1\}\}, \quad (4.36)$$

and

$$Q(x, de) = \delta_{\beta(x)}(de) \quad \text{a.s.} \quad (4.37)$$

Then  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with decomposition  $X = X^i + X^p$ , with

$$X_t^i = \int_{]0, t] \times \mathbb{R}} (e - X_{s-}) (\mu - \nu)(ds de), \quad (4.38)$$

$$X_t^p = X_0 + \int_0^t h(X_s) ds + \int_{]0, t] \times \mathbb{R}} \left( \int_{\mathbb{R}} (e - X_{s-}) Q(X_{s-}, de) \right) (\lambda(X_{s-}) ds + dp_s^*). \quad (4.39)$$

In particular  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = (e - X_{s-}(\omega)) \mathbb{1}_{\{X_{s-}(\omega) \in ]0, 1[ \}}(\omega, s)$ .

*Proof.* Let us prove that Hypothesis 3.17-(i) holds. We recall that the measure  $\mu$  was characterized by (4.29). We define  $\mu^c := \mu \mathbb{1}_{J^c}$ , and  $\nu^c := \nu \mathbb{1}_{J^c}$ .  $\nu^c$  is the compensator of  $\mu^c$ , see paragraph b) in [19]. Taking into account (4.31), (4.33) and (4.34), we have

$$\nu^c(ds de) = \lambda(X_{s-}) Q(X_{s-}, de) ds. \quad (4.40)$$

By Remark 3.19-(ii) we see that  $D \cap J^c = \cup_n [[T_n^i]]$ ,  $(T_n^i)_n$  totally inaccessible times. On the other hand, since by (4.34)  $J = K$ , we have  $D = K \cup (D \cap J^c)$ , therefore Hypothesis 3.17-(i) holds.

Let now consider Hypothesis 3.17-(ii). Taking into account (4.35), we have to prove that for every predictable time  $S$  such that  $[[S]] \subset \{(\omega, t) : X_{t-}(\omega) \in \{0, 1\}\}$ ,

$$\nu(\{S\}, de) = \mu(\{S\}, de) \quad \text{a.s.} \quad (4.41)$$



Let  $S$  be a predictable time satisfying  $[[S]] \subset \{(\omega, t) : X_{t-}(\omega) \in \{0, 1\}\}$ . By (4.29),  $\mu(\{S\}, de) = \delta_{X_S}(de)$ , while from (4.31) we get  $\nu(\{S\}, de) = Q(X_{S-}, de)$ . Therefore identity (4.41) can be rewritten as

$$Q(X_{S-}, de) = \delta_{X_S}(de) \quad \text{a.s.} \quad (4.42)$$

Previous identity holds true under assumptions (4.36) and (4.37), and so Hypothesis 3.17-(ii) is established.

In order to prove the validity of Hypothesis 4.2, we will make use of Lemma 3.22. We recall that the process  $X$  satisfies the stochastic differential equation (4.30), which gives, taking into account Lemma 4.8,

$$\begin{aligned} X_t = X_0 &+ \int_0^t h(X_s) ds + \int_{]0, t]} \left( \int_{\mathbb{R}} (e - X_s) Q(X_s, de) \right) \lambda(X_s) ds \\ &+ \int_{]0, t]} (\beta(X_{s-}) - X_{s-}) dp_s^* + \int_{]0, t] \times \mathbb{R}} (e - X_{s-}) (\mu - \nu)(ds de). \end{aligned} \quad (4.43)$$

We can show that previous equation is a particular case of (3.27). Indeed, we recall that, by (4.32) and (4.35), the support of the measure  $dp^*$  is included in  $K$ . We set  $B_s = s + p^*(s)$  and  $b(s, x) = (h(x) + \int_{\mathbb{R}} (e - x) \lambda(x) Q(x, de)) \mathbb{1}_{K^c}(s) + (\beta(x) - x) \mathbb{1}_K(s)$ . The reader can easily show that the sum of the first, second, and third integral in the right hand-side of (4.43) equals  $\int_0^t b(s, X_{s-}) dB_s$ , provided we show that  $\int_0^T |b(s, X_{s-})| dB_s$  is finite a.s. In fact we have

$$\begin{aligned} &\int_0^t |b(s, X_{s-})| dB_s \\ &\leq \int_0^t |h(X_s)| ds + \int_{]0, t]} \left| \int_{\mathbb{R}} (e - X_{s-}) \lambda(X_{s-}) Q(X_{s-}, de) \mathbb{1}_{K^c}(s) + (\beta(X_{s-}) - X_{s-}) \mathbb{1}_K(s) \right| dB_s \\ &= \int_0^t |h(X_s)| ds + \int_{]0, t]} \left| \int_{\mathbb{R}} (e - X_{s-}) Q(X_{s-}, de) (\lambda(X_{s-}) \mathbb{1}_{K^c}(s) + \mathbb{1}_K(s)) \right| (ds + dp^*(s)) \\ &\leq \int_0^t |h(X_s)| ds + \int_{]0, t]} \int_{\mathbb{R}} |e - X_{s-}| \nu(ds, de). \end{aligned} \quad (4.44)$$

Recalling Lemma 4.8, and taking into account that  $h$  is locally bounded, we get that  $\int_0^\cdot |b(s, X_{s-})| dB_s$  belongs to  $\mathcal{A}_{\text{loc}}^+$ . Then, setting  $N_s = 0$  and  $\gamma(s, x, e) = e - x$ , we see finally that  $X$  is a solution to equation (3.27).

Then, by Lemma 3.22,  $X$  satisfies Hypothesis 3.2, with decomposition  $X = X^i + X^p$ , where  $X^i$  and  $X^p$  are given respectively by (4.38) and (4.39). Moreover, the process  $X^i$  fulfills Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = (e - X_{s-}(\omega)) (1 - \mathbb{1}_K(\omega, s)) = (1 - \mathbb{1}_K(\omega, s)) \mathbb{1}_{\{X_{s-}(\omega) \in ]0, 1[ \}}(\omega, s)$ , and the process  $X^p$  satisfies Hypothesis 3.14.  $\square$

## 4.2 Identification of the BSDE's solution

We consider the following assumption on a couple  $(X, Y)$  of adapted processes.

**Hypothesis 4.10.**  *$X$  is a special weak Dirichlet process of finite quadratic variation, satisfying condition (2.7).  $Y_t = v(t, X_t)$  for some (deterministic) function  $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^{0,1}$  such that  $F = v$  and  $X$  verify condition (2.9).*

Let us remark the following facts.

**Proposition 4.11.** Assume that  $X$  is a process satisfying Hypothesis 3.2, with decomposition  $X = X^i + X^p$ , where  $X^i$  (resp.  $X^p$ ) fulfills Hypothesis 3.10 (resp. Hypothesis 3.14), with respect to  $\mu$ , with corresponding  $\tilde{\gamma}$ . Let in addition  $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a function of class  $C^{0,1}$ .

(a) If  $\sum_{s \leq T} |\Delta X_s|^2 < \infty$  a.s., then

$$|v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})|^2 \mathbb{1}_{\{|\tilde{\gamma}(s, e)| \leq 1\}} * \mu \in \mathcal{A}_{\text{loc}}. \quad (4.45)$$

(b) If  $X$  and  $F = v$  satisfy conditions (2.7) and (2.9), then

$$|v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})| \mathbb{1}_{\{|\tilde{\gamma}(s, e)| > 1\}} * \mu \in \mathcal{A}_{\text{loc}}^+. \quad (4.46)$$

(c) If  $X$  and  $F = v$  satisfy conditions (2.7) and (2.9), and moreover  $\sum_{s \leq T} |\Delta X_s|^2 < \infty$  a.s., then

$$v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-}) \in \mathcal{G}_{\text{loc}}^1(\mu).$$

*Proof.* Item (a) follows by Proposition 2.3 and inequality (3.10) in Proposition 3.15, with  $\varphi(\omega, s, x) = |v(s, X_{s-}(\omega) + x) - v(s, X_{s-}(\omega))|^2 \mathbb{1}_{\{|x| \leq 1\}}$ , allowing  $\varphi$  also depending on  $\omega$ .

Item (b) is a consequence of (2.7) and (2.9) together with Remark 2.5-(ii) and inequality (3.10) in Proposition 3.15, with  $\varphi(\omega, s, x) = |v(s, X_{s-}(\omega) + x) - v(s, X_{s-}(\omega))| \mathbb{1}_{\{|x| > 1\}}$ , allowing  $\varphi$  also depending on  $\omega$ .

Finally, item (c) is a direct consequence of items (a), (b), and Remark 2.6, with  $\varphi(\omega, s, e) = v(s, X_{s-}(\omega) + \tilde{\gamma}(\omega, s, e)) - v(s, X_{s-}(\omega))$  and  $A = \{(\omega, s, e) : |\tilde{\gamma}(\omega, s, e)| > 1\}$ .  $\square$

**Proposition 4.12.** Let  $\mu$  satisfy Hypothesis 3.17. Let  $X$  be a process verifying Hypothesis 4.2 with decomposition  $X = X^i + X^p$ , where  $\tilde{\gamma}$  is the predictable process which relates  $\mu$  and  $X^i$  in agreement with Hypothesis 3.10. Let  $(Y, Z, U)$  be a solution to the BSDE (4.1) such that the pair  $(X, Y)$  satisfies Hypothesis 4.10 with corresponding function  $v$ . Let  $X^c$  denote the continuous local martingale  $M^c$  of  $X$  given in the canonical decomposition (2.12).

Then, the pair  $(Z, U)$  fulfills

$$Z_t = \partial_x v(t, X_t) \frac{d\langle X^c, M \rangle_t}{d\langle M \rangle_t} \quad d\mathbb{P} \, d\langle M \rangle_t\text{-a.e.}, \quad (4.47)$$

$$\int_{]0, t] \times \mathbb{R}} H_s(e) (\mu - \nu)(ds \, de) = 0, \quad \forall t \in ]0, T], \text{ a.s.}, \quad (4.48)$$

with

$$H_s(e) := U_s(e) - (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})). \quad (4.49)$$

If, in addition,  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ ,

$$\int_{]0, T] \times \mathbb{R}} |H_s(e) - \hat{H}_s \mathbb{1}_K(s)|^2 \nu(ds \, de) = 0 \quad \text{a.s.} \quad (4.50)$$

*Remark 4.13.* Since the pair  $(X, Y)$  in Proposition 4.12 satisfies Hypothesis 4.10, then  $X$  and  $v$  in the statement satisfy (2.7) and (2.9). By Proposition 4.11-(c) it follows that  $v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-}) \in \mathcal{G}_{\text{loc}}^1(\mu)$ . Since  $U \in \mathcal{G}_{\text{loc}}^2(\mu) \subset \mathcal{G}_{\text{loc}}^1(\mu)$ , this yields  $H \in \mathcal{G}_{\text{loc}}^1(\mu)$ .

*Proof.* By assumption,  $X$  is a special weak Dirichlet process satisfying condition (2.7), and  $F = v$  is a function of class  $C^{0,1}$  satisfying the integrability condition (2.9). So we are in the condition to apply Theorem 2.14 to  $v(t, X_t)$ . We get

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) (\mu^X - \nu^X)(ds dx) \\ &\quad + \int_{]0, t]} \partial_x v(s, X_s) dX_s^c + A^v(t), \end{aligned} \quad (4.51)$$

where  $A^v : C^{0,1} \rightarrow \mathbb{D}^{ucp}$  is a map such that, for every  $v \in C^{0,1}$ ,  $A^v$  is a predictable orthogonal process. We set

$$\varphi(s, x) := v(s, X_{s-} + x) - v(s, X_{s-}).$$

Since  $X$  is of finite quadratic variation and verifies (2.7), and  $X$  and  $F = v$  satisfy (2.9), by Proposition 2.3 and Remark 2.5-(ii), we see that the process  $\varphi$  verifies condition (3.12) with  $A = \{|x| > 1\}$ . Moreover  $\varphi(s, 0) = 0$ . Since  $\mu$  verifies Hypothesis 3.17 and  $X$  verifies Hypothesis 4.2, we can apply Proposition 3.20 to  $\varphi(s, x)$ . Identity (4.51) becomes

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})) (\mu - \nu)(ds de) \\ &\quad + \int_{]0, t]} \partial_x v(s, X_s) dX_s^c + A^v(t). \end{aligned} \quad (4.52)$$

At this point we recall that the process  $Y_t = v(t, X_t)$  fulfills the BSDE (4.1), which can be rewritten as

$$\begin{aligned} Y_t &= Y_0 + \int_{]0, t]} Z_s dM_s + \int_{]0, t] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de) \\ &\quad - \int_{]0, t]} \tilde{g}(s, Y_{s-}, Z_s) d\zeta_s - \int_{]0, t] \times \mathbb{R}} \tilde{f}(s, e, Y_{s-}, U_s(e)) \lambda(ds de). \end{aligned} \quad (4.53)$$

By Proposition 2.12 the uniqueness of decomposition (4.52) yields identity (4.48) and

$$\int_{]0, t]} Z_s dM_s = \int_{]0, t]} \partial_x v(s, X_s) dX_s^c. \quad (4.54)$$

In particular, from (4.54) we get

$$\begin{aligned} 0 &= \left\langle \int_{]0, t]} Z_s dM_s - \int_{]0, t]} \partial_x v(s, X_s) dX_s^c, M_t \right\rangle \\ &= \int_{]0, t]} Z_s d\langle M \rangle_s - \int_{]0, t]} \partial_x v(s, X_s) \frac{d\langle X^c, M \rangle_s}{d\langle M \rangle_s} d\langle M \rangle_s \\ &= \int_{]0, t]} \left( Z_s - \partial_x v(s, X_s) \frac{d\langle X^c, M \rangle_s}{d\langle M \rangle_s} \right) d\langle M \rangle_s, \end{aligned}$$

that gives identification (4.47).

If in addition we assume that  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ , the predictable bracket at time  $t$  of the purely discontinuous martingale in identity (4.48) is well-defined, and equals

$$\int_{]0, t] \times \mathbb{R}} |H_s(e) - \hat{H}_s \mathbb{1}_J(s)|^2 \nu(ds de) + \sum_{s \in ]0, t]} |\hat{H}_s|^2 (1 - \hat{\nu}_s(\mathbb{R})) \mathbb{1}_{J \setminus K}(s), \quad (4.55)$$

see Theorem B.22, identity (B.25), and Remark B.23. The conclusion follows from the fact that under Hypothesis 3.17 we have  $J = K$ , see Remark 3.18.  $\square$

We apply now previous result to the case of Example 4.3. We start with a preliminary result.

**Lemma 4.14.** *Let  $\mu$  and  $X$  be respectively the Poisson random measure and the stochastic process satisfying the SDE (4.6) in Example 4.3. Let  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a function of  $C^{0,1}$  class such that  $x \mapsto \partial_x u(s, x)$  has linear growth, uniformly in  $s$ . Then condition (2.9) holds for  $X$  and  $F = u$ .*

*Proof.* We have

$$\begin{aligned}
& \int_{]0, \cdot] \times \mathbb{R}} |u(s, X_{s-} + x) - u(s, X_{s-}) - x \partial_x u(s, X_{s-})| \mathbb{1}_{\{|x| > 1\}} \mu^X(ds dx) \\
&= \sum_{0 < s \leq \cdot} |u(s, X_s) - u(s, X_{s-}) - \partial_x u(s, X_{s-}) \Delta X_s| \mathbb{1}_{\{|\Delta X_s| > 1\}} \\
&\leq \sum_{0 < s \leq \cdot} |\Delta X_s| \mathbb{1}_{\{|\Delta X_s| > 1\}} \left( \int_0^1 |\partial_x u(s, X_{s-} + a \Delta X_s)| da + \int_0^1 |\partial_x u(s, X_{s-})| da \right) \\
&\leq 2C \sum_{0 < s \leq \cdot} |X_{s-}| |\Delta X_s| \mathbb{1}_{\{|\Delta X_s| > 1\}} + \sum_{s \leq t} |\Delta X_s|^2 C \mathbb{1}_{\{|\Delta X_s| > 1\}} \\
&= 2C \int_{]0, \cdot] \times \mathbb{R}} |X_{s-}| |x| \mathbb{1}_{\{|x| > 1\}} \mu^X(ds dx) + \sum_{s \leq \cdot} |\Delta X_s|^2 \mathbb{1}_{\{|\Delta X_s| > 1\}}. \tag{4.56}
\end{aligned}$$

Since  $X$  is of finite quadratic variation, the second term in the right-hand side of (4.56) is in  $\mathcal{A}_{\text{loc}}^+$  if and only if

$$\sum_{s \in ]0, \cdot]} |\Delta X_s|^2 \in \mathcal{A}_{\text{loc}}^+, \tag{4.57}$$

see Proposition 2.2. Since by (4.6)  $\Delta X_s = \int_{\mathbb{R}} \gamma(X_{s-}, e) \mu(ds de)$ , we have

$$\sum_{s \in ]0, \cdot]} |\Delta X_s|^2 = \sum_{s \in ]0, \cdot]} \left| \int_{\mathbb{R}} \gamma(X_{s-}, e) \mu(ds de) \right|^2 = \int_{]0, \cdot] \times \mathbb{R}} |\gamma(X_{s-}, e)|^2 \mu(ds de),$$

and (4.57) reads

$$\int_{]0, \cdot] \times \mathbb{R}} |\gamma(X_{s-}, e)|^2 \mu(ds de) \in \mathcal{A}_{\text{loc}}^+. \tag{4.58}$$

Condition (4.58) holds because  $|\gamma(x, e)| \leq K(1 \wedge |e|)$  for every  $x \in \mathbb{R}$ ,  $\int_{\mathbb{R}} (1 \wedge |e|^2) \lambda(de) < \infty$  (see, respectively, (4.7) and (4.5)), and taking into account the fact that the integrand in (4.58) is locally bounded.

Finally, the first term in the right-hand side of (4.56) belongs to  $\mathcal{A}_{\text{loc}}^+$  since  $X_{s-}$  is locally bounded (see e.g. the lines above Theorem 15, Chapter IV, in [29]) and  $X$  satisfies (2.7). The conclusion follows.  $\square$

We are ready to give the identification result in the framework of Example 4.3.

**Corollary 4.15.** *Let  $(Y, Z, U) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{L}^2(\mu)$  be the unique solution to the BSDE (4.3). If the function  $u$  defined in (4.8) is of class  $C^{0,1}$  such that  $x \mapsto \partial_x u(t, x)$  has linear growth, uniformly in  $t$ , then the process  $(Z, U)$  satisfies*

$$Z_t = \partial_x u(t, X_t) \quad d\mathbb{P} dt\text{-a.e.}, \tag{4.59}$$

$$\int_{]0, t] \times \mathbb{R}} H_s(e) (\mu - \nu)(ds de) = 0, \quad \forall t \in ]0, T], \text{ a.s.} \tag{4.60}$$

where

$$H_s(e) := U_s(e) - (u(s, X_{s-} + \gamma(s, X_{s-}, e)) - u(s, X_{s-})). \quad (4.61)$$

If in addition  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ ,

$$U_s(e) = u(s, X_{s-} + \gamma(s, X_{s-}, e)) - u(s, X_{s-}) \quad d\mathbb{P} \lambda(de) ds\text{-a.e.} \quad (4.62)$$

*Proof.* We aim to apply Proposition 4.12. By Lemma 4.4,  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with decomposition  $X = X^i + X^p$ , where  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(s, e) = \gamma(s, X_{s-}, e)$ . Moreover, since  $X$  is a special semimartingale, it is of finite quadratic variation and (2.7) holds because of Corollary 11.26 in [18]. By Lemma 4.14, condition (2.9) holds for  $X$  and  $F = u$ , which implies that Hypothesis 4.10 is verified.

We can then apply Proposition 4.12: since  $X^c = M = W$ , (4.47) gives (4.59), while (4.48)-(4.49) with  $\tilde{\gamma}(s, e) = \gamma(s, X_{s-}, e)$  yield (4.60)-(4.61). If in addition  $H \in \mathcal{G}^2(\mu)$ , since  $\hat{H} = 0$  ( $\nu$  is absolutely continuous with respect to the Lebesgue measure), (4.50) yields

$$\int_{[0, T] \times \mathbb{R}} |H_s(e)|^2 \lambda(de) ds = 0, \quad (4.63)$$

and (4.62) follows.  $\square$

*Remark 4.16.* When the BSDE (4.3) is driven only by a standard Brownian motion, an identification result for  $Z$  analogous to (4.59) has been established by [17], even supposing only that  $f$  is Lipschitz with respect to  $Z$ .

Let us now consider a BSDE driven only by a purely discontinuous martingale, of the form (4.11). We formulate the following assumption for a couple of adapted processes  $(X, Y)$ .

**Hypothesis 4.17.** (i)  $Y = B + A'$ , with  $B$  a bounded variation process and  $A'$  a continuous orthogonal process;

(ii)  $Y_t = v(t, X_t)$  for some continuous deterministic function  $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , satisfying the integrability condition

$$\int_{[0, \cdot] \times \mathbb{R}} |v(t, X_{t-} + x) - v(t, X_{t-})| \mu^X(dt dx) \in \mathcal{A}_{\text{loc}}^+. \quad (4.64)$$

We have the following result.

**Proposition 4.18.** Let  $\mu$  satisfy Hypothesis 3.17. Let  $X$  verify Hypothesis 4.2 with decomposition  $X = X^i + X^p$ , where  $\tilde{\gamma}$  is the predictable process which relates  $\mu$  and  $X^i$  in agreement with Hypothesis 3.10. Let  $(Y, U)$  be a solution to the BSDE (4.11), such that  $(X, Y)$  satisfies Hypothesis 4.17 with corresponding function  $v$ .

Then, the process  $U$  satisfies

$$\int_{[0, t] \times \mathbb{R}} H_s(e) (\mu - \nu)(ds de) = 0 \quad \forall t \in ]0, T], \text{ a.s.}, \quad (4.65)$$

with

$$H_s(e) := U_s(e) - (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})). \quad (4.66)$$

If in addition  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ ,

$$\int_{[0, T] \times \mathbb{R}} |H_s(e) - \hat{H}_s \mathbb{1}_K(s)|^2 \nu(ds de) = 0 \quad \text{a.s.} \quad (4.67)$$

*Remark 4.19.* The assumption of continuity for  $v(t, x)$  in Hypothesis 4.17-(ii) is somehow restrictive since it can be relaxed with respect to  $x$ . However our purpose is to illustrate the methodology and the assumption of continuity simplifies the proof.

*Proof.* By assumption, the couple  $(X, Y)$  satisfies Hypothesis 4.17 with corresponding function  $v$ . We are then in the condition to apply Proposition 2.16 to  $v(t, X_t)$ . We get

$$v(t, X_t) = v(0, X_0) + \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) (\mu^X - \nu^X)(ds dx) + A^v(t), \quad (4.68)$$

where  $A^v$  is a predictable  $(\mathcal{F}_t)$ -orthogonal process. Set

$$\varphi(s, x) := v(s, X_{s-} + x) - v(s, X_{s-}).$$

By condition (ii) in Hypothesis 4.17, the process  $\varphi$  verifies condition (3.12) with  $A = \Omega \times [0, T] \times \mathbb{R}$ . Moreover  $\varphi(s, 0) = 0$ . Since  $\mu$  verifies Hypothesis 3.17, and  $X$  verifies Hypothesis 4.2 we can apply Proposition 3.20 to  $\varphi(s, x)$ . Identity (4.68) becomes

$$v(t, X_t) = v(0, X_0) + \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-} + \tilde{\gamma}(s, e)) - v(s, X_{s-})) (\mu - \nu)(ds de) + A^v(t). \quad (4.69)$$

At this point we recall that the process  $Y_t = v(t, X_t)$  fulfills the BSDE (4.11), which can be rewritten as

$$Y_t = Y_0 + \int_{]0, t] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de) - \int_{]0, t] \times \mathbb{R}} \tilde{f}(s, e, Y_{s-}, U_s(e)) \lambda(ds de). \quad (4.70)$$

By Proposition 2.12 the uniqueness of decomposition (4.69) yields identity (4.48). If in addition we assume that  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ , the predictable bracket at time  $t$  of the purely discontinuous martingale in identity (4.48) is well-defined, and equals

$$\int_{]0, t] \times \mathbb{R}} |H_s(e) - \hat{H}_s \mathbb{1}_J(s)|^2 \nu(ds de) + \sum_{s \in ]0, t]} |\hat{H}_s|^2 (1 - \hat{\nu}_s(\mathbb{R})) \mathbb{1}_{J \setminus K}(s), \quad (4.71)$$

see Theorem B.22, identity (B.25), and Remark B.23. The conclusion follows from the fact that under Hypothesis 3.17 we have  $J = K$ , see Remark 3.18.  $\square$

Previous result can be applied to the framework of Example 4.5. We start with a preliminary observation.

**Lemma 4.20.** *Let  $X, \mu$  be respectively the pure jump Markov process and the corresponding integer-valued random measure in Example 4.5. Let  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying (4.18), (4.19) and (4.20). If we set  $Y_t = u(t, X_t)$ , then  $(X, Y)$  satisfies Hypothesis 4.17 with corresponding function  $u$ .*

*Proof.* From (4.15) and the fact that  $u$  is continuous, it follows that

$$u(t, X_t) = u(0, X_0) + \sum_{s \leq t} (u(s, X_{s-} + \Delta X_s) - u(s, X_{s-})). \quad (4.72)$$

Obviously  $Y_t = u(t, X_t)$  has a finite number of jumps on each compact. We have  $\sum_{s \leq t} |u(s, X_{s-} + \Delta X_s) - u(s, X_{s-})| < \infty$  a.s. for every  $t \in \mathbb{R}_+$ . Therefore, condition (i) in Hypothesis 4.17 holds with  $B = u(0, X_0) + \sum_{s \leq \cdot} (u(s, X_{s-} + \Delta X_s) - u(s, X_{s-}))$ ,  $A' = 0$ .

To verify the validity of condition (ii) of Hypothesis 4.17 with corresponding function  $v = u$ , we have to show that (4.64) holds with  $v = u$ . Denoting  $\|\lambda\|_\infty = \sup_{t \in [0, T], x \in \mathbb{R}} |\lambda(t, x, \mathbb{R})|$ , by (4.14) we have

$$\begin{aligned} & \mathbb{E} \left[ \int_{]0, T] \times \mathbb{R}} |u(s, X_{s-} + x) - u(s, X_{s-})| \mu^X(ds dx) \right] \\ &= \mathbb{E} \left[ \int_{]0, T] \times \mathbb{R}} |u(s, e) - u(s, X_{s-})| \mu(ds de) \right] \\ &= \mathbb{E} \left[ \int_{]0, T] \times \mathbb{R}} |u(s, e) - u(s, X_{s-})| \lambda(s, X_{s-}, de) ds \right] \\ &\leq T \|\lambda\|_\infty^{1/2} \|u(s, e) - u(s, X_{s-})\|_{\mathcal{L}^2(\mu)}^{1/2} \end{aligned}$$

and the conclusion follows since  $u(s, e) - u(s, X_{s-}) \in \mathcal{L}^2(\mu)$  by (4.19).  $\square$

We have the following identification result in the framework of Example 4.5.

**Corollary 4.21.** *Let  $(Y, U) \in \mathcal{L}^2 \times \mathcal{L}^2(\mu)$  be the unique solution to the BSDE (4.12) and  $X, u$  respectively the process and the function appearing in Example 4.5. Assume moreover that  $u$  is continuous. Then the process  $U$  satisfies*

$$U_t(e) = u(t, e) - u(t, X_{t-}) \quad d\mathbb{P} \lambda(t, X_{t-}, de) dt\text{-a.e.} \quad (4.73)$$

*Proof.* We aim to apply Proposition 4.18. By Lemma 4.6,  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with decomposition  $X = X^i$ ,  $X^p = 0$ , where  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(s, e) = e - X_{s-}$ . Moreover, by Lemma 4.20,  $(X, Y)$  satisfies Hypothesis 4.17 with corresponding function  $v = u$ . We can then apply Proposition 4.18. We have

$$\begin{aligned} H_s(e) &:= U_s(e) - (u(s, X_{s-} + \tilde{\gamma}(s, e)) - u(s, X_{s-})) \\ &= U_s(e) - (u(s, e) - u(s, X_{s-})), \end{aligned} \quad (4.74)$$

which belongs to  $\mathcal{L}^2(\mu)$ , and therefore to  $\mathcal{G}^2(\mu)$ . Since moreover  $\hat{H} = 0$  ( $\nu$  is absolutely continuous with respect to the Lebesgue measure), (4.67) yields

$$\int_{]0, T] \times \mathbb{R}} |H_s(e)|^2 \lambda(s, X_{s-}, de) ds = 0, \quad \text{a.s.} \quad (4.75)$$

and (4.73) follows.  $\square$

Finally, we apply previous results to Example 4.7.

**Lemma 4.22.** *Let  $(Y, U) \in \mathcal{L}^2 \times \mathcal{G}^2(\mu)$  be a solution to the BSDE (4.24) and  $X, u$  respectively the process and the function appearing in Example 4.7. Assume that  $Y_t = u(t, X_t)$  for some continuous function  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ . Then  $(X, Y)$  satisfies Hypothesis 4.17 with corresponding function  $v = u$ .*

*Proof.* Since the process  $X$  has a finite number of jumps on each compact, the same holds for  $Y_t = u(t, X_t)$ . We set

$$B_t := \sum_{0 < s \leq t} \Delta Y_s, \quad A'_t := Y_t - B_t. \quad (4.76)$$



Obviously  $B$  has bounded variation, and the process  $A'$  is continuous by definition. Since  $Y$  satisfies by assumption BSDE (4.24), for every local continuous martingale  $N$  we have

$$[Y, N]_t = \int_{]0, t]} f(s, X_{s-}, Y_{s-}, U_s(\cdot)) d[A, N]_s - \left[ \int_{]0, \cdot] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de), N \right]_t. \quad (4.77)$$

Since  $A$  is a predictable increasing process, therefore has bounded variation,  $[A, N] = 0$  by Proposition 3.13 in [2]. The second term in (4.77) is zero because  $\int_{]0, \cdot] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de)$  is a purely discontinuous martingale. Therefore (4.77) vanishes. Recalling that  $B$  has bounded variation, it also follows that  $[B, N] = 0$ , so that  $A'$  is a continuous  $(\mathcal{F}_t)$ -orthogonal process, and condition (i) in Hypothesis 4.17 holds.

It remains to show that  $u(t, X_t)$  satisfies condition (4.64) with  $v = u$ . Since  $u$  is continuous, we have

$$\int_{]0, \cdot] \times \mathbb{R}} |u(s, X_{s-} + x) - u(s, X_{s-})| \mu^X(ds dx) = \sum_{0 < s \leq \cdot} |u(s, X_s) - u(s, X_{s-})| = \sum_{s \leq \cdot} |\Delta Y_s|. \quad (4.78)$$

The process  $Y$  takes values in the image of  $[0, T] \times [0, 1]$  with respect to  $u$ , which is a compact set. Therefore the jumps of  $Y$  are bounded, and (4.78) belongs to  $\mathcal{A}_{\text{loc}}^+$ , see for instance the proof of Corollary in pag 373 in [29].

□

**Corollary 4.23.** *Let  $(Y, U) \in \mathcal{L}^2 \times \mathcal{G}^2(\mu)$  be a solution to the BSDE (4.24), and  $X$  the piece-wise deterministic Markov process with local characteristics  $(h, \lambda, Q)$  appearing in Example 4.7. Assume that  $Y_t = u(t, X_t)$  for some continuous function  $u$ . Assume in addition that there exists a function  $\beta : \{0, 1\} \rightarrow \mathbb{R}$ , such that*

$$X_s = \beta(X_{s-}) \quad \text{on } \{(\omega, s) : X_{s-}(\omega) \in \{0, 1\}\}, \quad (4.79)$$

and

$$Q(x, de) \mathbb{1}_{\{x \in \{0, 1\}\}}(s) = \delta_{\beta(x)}(de). \quad (4.80)$$

Then the process  $U$  satisfies

$$\int_{]0, t] \times \mathbb{R}} H_s(e) (\mu - \nu)(ds de) = 0 \quad \forall t \in ]0, T], \text{ a.s.}, \quad (4.81)$$

where

$$H_s(e) := (U_s(e) - (u(s, e) - u(s, X_{s-})) \mathbb{1}_{\{X_{s-} \in (0, 1)\}}(s) + U_s(e) \mathbb{1}_{\{X_{s-} \in \{0, 1\}\}}(s).$$

If in addition  $H_s(e) \in \mathcal{G}_{\text{loc}}^2(\mu)$ ,

$$U_s(e) = u(s, e) - u(s, X_{s-}) \quad d\mathbb{P} \lambda(X_{s-}) Q(X_{s-}, de) ds\text{-a.e.} \quad (4.82)$$

*Remark 4.24.* If  $H \in \mathcal{G}_{\text{loc}}^2(\mu)$ , the value of  $U_s(\cdot)$  can be chosen on  $K = \{(\omega, s) : X_{s-}(\omega) \in \{0, 1\}\}$  as an arbitrary  $\mathcal{P}$ -measurable process, see Proposition B.28.

*Proof.* We will apply Proposition 4.18. By Lemma 4.9,  $\mu$  satisfies Hypothesis 3.17 and  $X$  fulfills Hypothesis 4.2 with decomposition  $X = X^i + X^p$ , where  $X^i$  satisfies Hypothesis 3.10 with  $\tilde{\gamma}(\omega, s, e) = (e - X_{s-}(\omega)) \mathbb{1}_{\{X_{s-}(\omega) \in ]0, 1\}}(\omega, s)$ . Moreover, by Lemma 4.22, Hypothesis 4.17 holds for  $(X, Y)$ . We are then in condition to apply Proposition 4.18. Identity (4.65) yields

$$\int_{]0, t] \times \mathbb{R}} H_s(e) (\mu - \nu)(ds de) = 0 \quad \forall t \in [0, T], \text{ a.s.}, \quad (4.83)$$

where

$$\begin{aligned}
H_s(e) &:= U_s(e) - (u(s, X_{s-} + \tilde{\gamma}(s, e)) - u(s, X_{s-})) \\
&= U_s(e) - (u(s, X_{s-} + (e - X_{s-}) \mathbb{1}_{\{X_{s-} \in ]0,1[ \}}(s)) - u(s, X_{s-})) \\
&= (U_s(e) - (u(s, e) - u(s, X_{s-})) \mathbb{1}_{\{X_{s-} \in ]0,1[ \}}(s) + U_s(e) \mathbb{1}_{\{X_{s-} \in \{0,1\} \}}(s), \\
&= (U_s(e) - (u(s, e) - u(s, X_{s-})) \mathbb{1}_{K^c}(s) + U_s(e) \mathbb{1}_K(s),
\end{aligned} \tag{4.84}$$

where in the latter equality we use the fact that  $K = \{(\omega, s) : X_{s-}(\omega) \in \{0,1\}\}$ .

It remains to prove (4.82). We recall that  $\nu^c := \nu \mathbb{1}_{J^c}$  verifies  $\nu^c(ds de) = \lambda(X_s) Q(X_s, de) ds$  by (4.40). We set  $\nu^d := \nu \mathbb{1}_J$ ; since  $J = K$ , we have

$$\nu^d(ds de) = \nu(ds de) \mathbb{1}_K(s) = Q(X_{s-}, de) dp_s^* = \delta_{\beta(X_{s-})}(de) dp_s^*. \tag{4.85}$$

If  $H_s(e)$  belongs to  $\mathcal{G}_{\text{loc}}^2(\mu)$ , recalling identity (B.32) in Remark B.23, identity (4.67) and (4.84) yield

$$\begin{aligned}
0 &= \int_{]0, T] \times \mathbb{R}} |H_s(e)|^2 \nu^c(ds de) + \int_{]0, T] \times \mathbb{R}} |H_s(e) - \hat{H}_s \mathbb{1}_K(s)|^2 \nu^d(ds de) \\
&= \int_{]0, T] \times \mathbb{R}} |U_s(e) - (u(s, e) - u(s, X_{s-}))|^2 \nu^c(ds de) + \int_{]0, T] \times \mathbb{R}} |U_s(e) - \hat{U}_s \mathbb{1}_K(s)|^2 \nu^d(ds de).
\end{aligned} \tag{4.86}$$

Taking into account condition (4.85), (4.33) and (4.35), we have

$$\hat{U}_s \mathbb{1}_K(s) = \int_{\mathbb{R}} U_s(e) \nu^d(\{s\} de) = \int_{\mathbb{R}} U_s(e) \delta_{\beta(X_{s-})}(de) \mathbb{1}_K(s) = U_s(\beta(X_{s-})) \mathbb{1}_K(s).$$

Consequently

$$\int_{]0, T] \times \mathbb{R}} |U_s(e) - \hat{U}_s \mathbb{1}_K(s)|^2 \nu^d(ds de) = \int_{]0, T] \times \mathbb{R}} |U_s(e) - \hat{U}_s \mathbb{1}_K(s)|^2 \delta_{\beta(X_{s-})}(de) dp_s^* = 0.$$

Therefore (4.86) gives simply

$$0 = \int_{]0, T] \times \mathbb{R}} |U_s(e) - (u(s, e) - u(s, X_{s-}))|^2 \lambda(X_s) Q(X_s, de) ds,$$

and (4.82) follows.  $\square$

*Remark 4.25.* In all the considered examples, the underlying process  $X$  was a Markov process which is a semimartingale. However, in the literature there are plenty of examples that are not semimartingales, even in the continuous case.

Let  $X$  be a solution of an SDE with distributional drift, see e.g. [16, 30, 15], of the type

$$dX_t = \beta(X_t) dt + dW_t, \tag{4.87}$$

for a class of Schwartz distributions  $\beta$ . In particular in the one-dimensional case  $\beta$  is allowed to be the derivative of any continuous function. In this case  $X$  is not a semimartingale but only a Dirichlet process, so that, for  $v \in C^{0,1}$ ,  $v(t, X_t)$  is a weak Dirichlet process. FBSDEs related to a forward process  $X$  solving (4.87) have been studied for instance in [31], when the terminal type is random.

## Appendix

In what follows we refer to the notations introduced at the beginning of Section 2.  $(\mathcal{F}_t)_{t \in [0, T]}$  will be a fixed filtration fulfilling the usual conditions, and it will be often omitted. A random set will be a subset of  $\Omega \times [0, T] \cup \{\infty\}$ , and  $[[\tau, \tau']]$  will denote the stochastic interval  $\{(\omega, t) : t \in [0, T] \cup \{\infty\}, \tau(\omega) \leq t \leq \tau'(\omega)\}$  associated to two stopping times  $\tau, \tau'$ . For a stopping time  $\tau$  taking values in  $[0, T] \cup \{\infty\}$ ,  $\mathcal{F}_{\tau-}$  will denote the  $\sigma$ -field generated by  $\mathcal{F}_0$  and the events  $A \cap \{t < \tau\}$ , where  $t \in [0, T]$  and  $A \in \mathcal{F}_t$ , see (0.30) of [20]. In the sequel, a random set will be called predictable (resp. optional) if its restriction to  $\Omega \times [0, T]$  is  $\mathcal{P}$ -measurable (resp.  $\mathcal{O}$ -measurable). Analogously, a stochastic process which is  $\mathcal{P}$ -measurable (resp.  $\mathcal{O}$ -measurable) will be called predictable (resp. optional).

### A General theory of Stochastic Processes

**Definition A.1** (Definition 1.30, Chapter I, in [21]). *A random set  $A$  is called to be thin if it is of the form  $A = \cup_n [[T_n]]$ , where  $(T_n)$  is a sequence of stopping times; if moreover the sequence  $(T_n)$  satisfies  $[[T_n]] \cap [[T_m]] = \emptyset$  for all  $n \neq m$ , it is called an exhausting sequence for  $A$ .*

*Remark A.2.* Any optional random set whose sections are at most countable is thin in the sense of Definition A.1, see the comments below Definition 1.30, Chapter I, in [21].

**Definition A.3** (Definition 1.15, [18]). *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $\mathcal{G}$  be a sub- $\sigma$ -field of  $\mathcal{F}$ . A random variable  $\xi$  is called to be  $\sigma$ -integrable with respect to  $\mathcal{G}$  if there exists  $\Omega_n \in \mathcal{G}$ ,  $\Omega_n \uparrow \Omega$  a.s. such that each  $\xi \mathbb{1}_{\Omega_n}$  is integrable.*

**Definition A.4** (Definition 7.39 in [18]). *An optional process  $X = (X_t)$  is said to be thin if  $\{\Delta X \neq 0\}$  is a thin set. A typical example of thin optional process is the jump  $\Delta X$  of an adapted càdlàg process  $X$ .*

**Definition A.5** (Definition 7.33, in [18]). *Let  $M$  and  $N$  be two local martingales. If  $[M, N] = 0$ , we say that  $M$  and  $N$  are mutually orthogonal.*

The notion of purely discontinuous martingales appears for instance Definition 7.21, in [18]. Below we recall a useful characterization of such processes given in Theorem 7.34, in [18], the comments above and obvious localization arguments.

**Theorem A.6.** *Let  $M$  be a local martingale with  $M_0 = 0$ . Then  $M$  is purely discontinuous if and only if it is orthogonal to every continuous local martingale.*

**Definition A.7** (Definition 1.10, Chapter I, in [21]). *A random set  $A$  is called evanescent if the set  $\{\omega : \exists t \in [0, T] \cup \{\infty\} \text{ with } (\omega, t) \in A\}$  is  $\mathbb{P}$ -null; two  $E$ -valued processes are called indistinguishable if the random set  $\{X \neq Y\} = \{(\omega, t) : X_t(\omega) \neq Y_t(\omega)\}$  is evanescent, i.e., if almost all the paths of  $X$  and  $Y$  are the same.*

**Theorem A.8** (Theorem 4.18, Chapter I, in [21]). *Any local martingale  $M$  admits a unique (up to indistinguishability) decomposition*

$$M = M^c + M^d$$

where  $M_0^d = 0$ ,  $M^c$  is a continuous local martingale and  $M^d$  is a purely discontinuous local martingale.

In the sequel  $\mathcal{H}^{2,d}$  (resp.  $\mathcal{H}_{\text{loc}}^{2,d}$ ) will stand for the set of square integrable (resp. locally square integrable) purely discontinuous martingales.

**Corollary A.9** (Corollary 4.19, Chapter I, in [21]). *Let  $M$  and  $N$  be two purely discontinuous local martingales having the same jumps  $\Delta M = \Delta N$  (up to an evanescent set). Then  $M$  and  $N$  are indistinguishable.*

**Proposition A.10** (Proposition 2.4-(a) and Proposition 2.6, Chapter I, in [21]). *If  $X$  is a predictable process, then  $\Delta X$  is predictable. If moreover  $\tau$  is a stopping time, then  $X_\tau \mathbb{1}_{\{\tau < \infty\}}$  is  $\mathcal{F}_{\tau-}$ -measurable.*

## A.1 Predictable and totally inaccessible stopping times, predictable projection

**Definition A.11** (Definition 2.7, Chapter I, in [21]). *A predictable time is a mapping  $\tau : \Omega \rightarrow [0, T] \cup \{\infty\}$ , such that the stochastic interval  $[[0, \tau[$  is predictable.*

*Remark A.12.* If  $\tau$  is a predictable (finite) time, then  $[[\tau]] \in \mathcal{P}$ , see e.g. the comments after Definition 2.7, Chapter I, in [21].

**Proposition A.13** (Proposition 2.18-(b), Chapter I, in [21]). *If  $X$  and  $Y$  are two predictable processes satisfying  $X_\tau = Y_\tau$  a.s. on  $\{\tau < \infty\}$  for all predictable times  $\tau$ , then  $X$  and  $Y$  are indistinguishable.*

**Definition A.14** (Definition 2.20, Chapter I, in [21]). *A stopping time  $\tau$  is called totally inaccessible if  $\mathbb{P}(\tau = S < \infty) = 0$  for all predictable time  $S$ .*

*Remark A.15.* It straight follows from Definition A.14 that

$$\mathbb{1}_{[[T^i]]}(\omega, T^p(\omega)) \mathbb{1}_{\{T^i < \infty, T^p < \infty\}} = 0 \quad \text{a.s.} \quad (\text{A.1})$$

for any totally inaccessible time  $T^i$  and predictable time  $T^p$ .

Indeed, taking the expectation of the left-hand side of (A.1) we get

$$\mathbb{E} [\mathbb{1}_{[[T^i]]}(\cdot, T^p(\cdot)) \mathbb{1}_{\{T^i < \infty, T^p < \infty\}}] = \mathbb{P}(\omega \in \Omega : T^i(\omega) = T^p(\omega) < \infty) = 0.$$

**Lemma A.16** (Lemma 2.23, Chapter I, in [21]). *If  $A$  is a predictable thin set, then  $A$  admits an exhausting sequence of predictable times, namely there is a sequence  $(T_n)$  of predictable times whose graphs are pairwise disjoint, such that  $A = \cup_n [[T_n]]$ .*

**Proposition A.17** (Proposition 2.24, Chapter I, in [21]). *If  $X$  is a càdlàg predictable process, there is a sequence of predictable times that exhausts the jumps of  $X$ . Furthermore,  $\Delta X_\tau = 0$  a.s. on  $\{\tau < \infty\}$  for all totally inaccessible time  $\tau$ .*

**Definition A.18** (Definition 2.25, Chapter I, in [21]). *A càdlàg process  $X$  is quasi-left continuous if  $\Delta X_\tau = 0$  a.s. on the set  $\{\tau < \infty\}$  for every predictable time  $\tau$ .*

**Proposition A.19** (Proposition 2.26, Chapter I, in [21]). *Let  $X$  be a càdlàg adapted process.  $X$  is quasi-left continuous if and only if there is a sequence of totally inaccessible times that exhausts the jumps of  $X$ .*

**Theorem A.20** (Theorem 4.21, Chapter IV, [18]). *For any adapted càdlàg process  $X = (X_t)$  there exists a sequence  $(T_n)_n$  of strictly positive stopping times satisfying the following conditions:*

- (i)  $\{\Delta X \neq 0\} \subset \cup_n [[T_n]]$ ;
- (ii) each  $T_n$  is predictable or totally inaccessible;
- (iii)  $[[T_n]] \cap [[T_m]] = \emptyset$  for every  $m \neq n$ .

**Theorem A.21** (Theorem 5.2, [18]). *Let  $X$  be a measurable process such that for every predictable time  $\tau$ ,  $X_\tau$  is  $\sigma$ -integrable with respect to  $\mathcal{F}_{\tau-}$ . Then there exists a unique predictable process, called predictable projection, denoted by  ${}^pX$ , such that for every predictable time  $\tau$  we have*

$$\mathbb{E} [X_\tau \mathbb{1}_{\{\tau < \infty\}} | \mathcal{F}_{\tau-}] = {}^pX_\tau \mathbb{1}_{\{\tau < \infty\}} \quad \text{a.s.}$$

**Lemma A.22** (Lemma 1.37 in [20]). *Let  $A$  be an increasing predictable process with  $A_0 = 0$ . Then there exists a sequence of increasing stopping times  $(T_n)$ , such that,  $T_n(\omega) \uparrow +\infty$ , and  $A_{T_n \wedge T} \leq n$  for each  $n$ .*

**Lemma A.23.** *Let  $A$  be a predictable process such that  $\sup_{t \leq u} |A_t| < \infty$  a.s.  $\forall u > 0$ . Then, for every predictable time  $\tau$  taking values in  $]0, T] \cup \{+\infty\}$ , we have that  $A_\tau \mathbb{1}_{\{\tau < \infty\}}$  is  $\sigma$ -integrable with respect to  $\mathcal{F}_{\tau-}$ .*

*Proof.* We set  $A_t^* = \sup_{s \leq t} A_s$ .  $A^*$  is a predictable and increasing process. Moreover  $A_0 = 0$ . By Lemma A.22 there exists a sequence of stopping times  $(T_n)$ , such that  $T_n \uparrow \tau = \inf\{t : A_t^* = \infty\} = \infty$ , with  $A_{T_n}^* \leq n$  for each  $n$ . Let  $\Omega_n = \{T_n \geq \tau\} \cap \{\tau < \infty\}$ . Clearly  $\cup_n \Omega_n = \{\tau < \infty\}$ . Moreover

$$n \geq A_\tau^* \mathbb{1}_{\Omega_n} \in L^1.$$

By Theorem 56, Chapitre IV, in [11],  $\Omega_n \in \mathcal{F}_{\tau-}$ , so the result follows.  $\square$

**Corollary A.24.** *Let  $A$  be a predictable process such that  $\sup_{t \leq u} |A_t| < \infty$  a.s.  $\forall u \in [0, T]$ . Then its predictable projection exists and  ${}^pA = A$ .*

*Proof.* Let  $\tau$  be a predictable time. By (1.5) in [20],  $A_\tau \mathbb{1}_{\{\tau < \infty\}}$  is  $\mathcal{F}_{\tau-}$ -measurable. This, together with Lemma A.23, gives

$$\mathbb{E} [A_\tau \mathbb{1}_{\{\tau < \infty\}} | \mathcal{F}_{\tau-}] = A_\tau \mathbb{1}_{\{\tau < \infty\}} \quad \text{a.s.}$$

From Theorem A.21 we conclude that  ${}^pA = A$ .  $\square$

**Definition A.25** (Definition 2.32, Chapter I, in [21]). *A random set  $A$  is called measurable if its restriction to  $\Omega \times [0, T]$  is measurable. The predictable support of a measurable random set  $A$  is the predictable set  $A' = \{^p(\mathbb{1}_A) > 0\}$ , which is defined up to an evanescent set.*

**Proposition A.26** (Proposition 2.35, Chapter I, in [21]). *Let  $X$  be a càdlàg adapted process.  $X$  is quasi-left continuous if and only if the predictable support of the random set  $\{\Delta X \neq 0\}$  is evanescent.*

**Remark A.27.** For any totally inaccessible time  $T^i$  we have

$${}^p(\mathbb{1}_{[[T^i]]} \mathbb{1}_{\{T^i < \infty\}}) = 0.$$

Indeed, by Theorem A.21, for every predictable time  $\tau$ , we have

$${}^p(\mathbb{1}_{[[T^i]]}(\tau) \mathbb{1}_{\{T^i < \infty\}}) \mathbb{1}_{\{\tau < \infty\}} = \mathbb{E} [\mathbb{1}_{[[T^i]]}(\tau) \mathbb{1}_{\{T^i, \tau < \infty\}} | \mathcal{F}_{\tau-}]$$

which vanishes since  $\mathbb{1}_{[[T^i]]}(\tau) \mathbb{1}_{\{T^i, \tau < \infty\}} = 0$  a.s., see Remark A.15.

As we will see in the next section, the notion of predictable projection for a measurable process plays a fundamental role in the stochastic integration theory with respect to random measures. We have the following important result.

**Theorem A.28** (Theorem 4.56, point c), Chapter I, in [21]). *Let  $H$  be an optional process with  $H_0 = 0$ . We have  ${}^pH = 0$  and  $[\sum_{s \leq \cdot} |H_s|^2]^{1/2} \in \mathcal{A}_{\text{loc}}^+$  if and only if there exists a local martingale  $M$  such that  $\Delta M$  and  $H$  are indistinguishable.*

## B Random measures

In the present section some basic results on stochastic integration with respect to (nonnegative) random measures are recalled. By default, excepted if the contrary is explicitly mentioned, all the considered measures will be non-negative. These results are presented without proof, for a complete discussion on this topic see, e.g., Chapter II, Section 1, in [21], or Chapter XI, Section 1, in [18].

In what follows  $(E, \mathcal{E})$  will be the measurable space constituted by  $E = \mathbb{R}$  and its Borel  $\sigma$ -algebra  $\mathcal{E}$ . We remark however that the mentioned references consider the case when  $(E, \mathcal{E})$  is any Blackwell space.

### B.1 General random measures

**Definition B.1** (Definition 1.3, Chapter II, in [21]). *A random measure on  $[0, T] \times E$  is a family  $\mu = (\mu(\omega, dt de) : \omega \in \Omega)$  of measures on  $([0, T] \times E, \mathcal{B}([0, T]) \otimes \mathcal{E})$  satisfying the following.*

1. *For every  $A \in \mathcal{B}([0, T]) \otimes \mathcal{E}$ , the map  $\omega \mapsto \mu(\omega, A)$  is a (measurable) random variable.*
2.  *$\mu(\omega, \{0\} \times E) = 0$  identically.*

Let  $\mu$  be a random measure and  $W \in \tilde{\mathcal{O}}$ . Since  $(t, e) \mapsto W_t(\omega, e)$  is  $\mathcal{B}([0, T]) \otimes \mathcal{E}$ -measurable for each  $\omega \in \Omega$ , we can define the integral process  $W * \mu$  by

$$W * \mu_t(\omega) = \int_{]0, t] \times E} W_s(\omega, e) \mu(\omega, ds de).$$

*Remark B.2.* We remark that for fixed  $\omega$ , previous integral is a Lebesgue type integral. When  $W$  is positive (resp. negative), previous integral always exists but could be  $+\infty$  (resp.  $-\infty$ ).

In the sequel, given a random measure  $\mu$  as before, we will often omit the reference to  $\omega$ . In other words, we will write  $\mu(dt de)$  instead of  $\mu(\omega, dt de)$ .

**Definition B.3** (Definition 1.6, Chapter II, in [21]). *(a) A random measure  $\mu$  is called optional if the process  $W * \mu$  is  $\mathcal{O}$ -measurable for every  $W \in \tilde{\mathcal{O}}$ . A random measure  $\lambda$  is called predictable if the process  $W * \lambda$  is  $\mathcal{P}$ -measurable for every  $W \in \tilde{\mathcal{P}}$ .*

*(b) An optional random measure  $\mu$  is called integrable if  $1 * \mu \in \mathcal{A}^+$ .*

*(c) An optional random measure  $\mu$  is called  $\tilde{\mathcal{P}}$ - $\sigma$ -finite if there exists a  $\tilde{\mathcal{P}}$ -measurable partition  $(A_n)$  of  $\tilde{\Omega}$  such that each  $1_{A_n} * \mu \in \mathcal{A}^+$ .*

**Theorem B.4** (Theorem 1.8, Chapter II, in [21]). *Let  $\mu$  be an optional  $\tilde{\mathcal{P}}$ - $\sigma$ -finite random measure. There exists a random measure, called the compensator of  $\mu$  and denoted by  $\nu$ , which is unique up to a  $\mathbb{P}$ -null set, and which is characterized as being a predictable random measure satisfying*

$$\mathbb{E}[W * \nu_T] = \mathbb{E}[W * \mu_T],$$

*for every nonnegative  $W \in \tilde{\mathcal{P}}$ . Moreover, there exists a predictable process  $A \in \mathcal{A}^+$  and a kernel  $\phi(\omega, t, de)$  from  $(\Omega \times [0, T], \mathcal{P})$  into  $(E, \mathcal{E})$  such that*

$$\nu(\omega, dt de) = dA_t(\omega) \phi(\omega, t, de). \tag{B.1}$$

*Of course, the disintegration (B.1) is not unique.*

## B.2 Integer-valued random measures

**Definition B.5** (Definition 1.13, Chapter II, in [21]). *An integer-valued random measure is a random measure that satisfies the following properties.*

- (i)  $\mu(\omega, \{t\} \times E) \leq 1$  identically;
- (ii) for each  $A \in [0, T] \times E$ ,  $\mu(\cdot, A)$  takes values in  $\mathbb{N}$ .
- (iii)  $\mu$  is optional and  $\tilde{\mathcal{P}}$ - $\sigma$ -finite.

**Proposition B.6** (Proposition 1.14, Chapter II, in [21]). *Let  $\mu$  be an integer-valued random measure. We set*

$$D = \{(\omega, s) \mid \mu(\omega, \{s\} \times E) = 1\}. \quad (\text{B.2})$$

*The random set  $D$  is thin ( $D$  is called the support of  $\mu$ ) and there exists an  $E$ -valued optional process  $\beta$  such that*

$$\mu(\omega, dt de) = \sum_{s \geq 0} \mathbb{1}_D(\omega, s) \delta_{(s, \beta_s(\omega))}(dt de). \quad (\text{B.3})$$

*Remark B.7.* Let  $\mu$  be an integer-valued random measure, with associated support  $D$  and process  $\beta$  in the sense of (B.3). Then, for any  $W \in \tilde{\mathcal{O}}$ , we have

$$W * \mu_t = \sum_{s \in ]0, t]} W_s(\beta_s) \mathbb{1}_D(s). \quad (\text{B.4})$$

**Proposition B.8** (Proposition 1.16, Chapter II, in [21]). *Let  $X = (X_t)$  be an adapted càdlàg  $E$ -valued process. Then*

$$\mu^X(\omega, dt dx) = \sum_{s \in ]0, T]} \mathbb{1}_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt dx) \quad (\text{B.5})$$

*defines an integer-valued random measure on  $[0, T] \times E$ , and in the representation (B.3) we have  $D = \{\Delta X \neq 0\}$  and  $\beta = \Delta X$ .*

**Corollary B.9** (Corollary 1.19, Section II, in [21]). *Let  $X$  be an adapted càdlàg process and  $\mu^X$  be the measure associated to its jumps by (B.5), and  $\nu^X$  its compensator. Then  $X$  is quasi-left continuous if and only if there exists a version of  $\nu^X$  that satisfies identically  $\nu^X(\omega, \{s\}, de) = 0$ .*

**Theorem B.10** (Theorem 11.14 in [18]). *Let  $\mu$  be the integer-valued random measure with support  $D$ , and let  $\nu$  be its compensator. Set*

$$a = (a_t), \quad a_t = \nu(\{t\} \times E), \quad t \geq 0, \quad (\text{B.6})$$

$$J = \{a > 0\}, \quad (\text{B.7})$$

$$K = \{a = 1\}. \quad (\text{B.8})$$

*Then  $a$  is a predictable thin process,  $0 \leq a \leq 1$ ,  $J$  is the predictable support of  $D$ , and  $K$  is the largest predictable set contained in  $D$  (up to an evanescent set).*

**Proposition B.11** (Proposition 1.17, Chapter II, in [21]). *Let  $\mu$  be an integer-valued random measure,  $\nu$  its compensator, and  $J = \{(\omega, t) : \nu(\omega, \{t\} \times E) > 0\}$ .*

- a)  $J$  is a predictable thin set.



b) For all predictable times  $\tau$  and nonnegative  $W \in \tilde{\mathcal{P}}$  (or, equivalently, for every  $W \in \tilde{\mathcal{P}}$  such that  $\int_E W(\tau, e) \mu(\{\tau\}, de) \mathbb{1}_{\{\tau < \infty\}}$  exists)

$$\int_E W_\tau(e) \nu(\{\tau\}, de) = \mathbb{E} \left[ \int_E W_\tau(e) \mu(\{\tau\}, de) \middle| \mathcal{F}_{\tau-} \right] \text{ on } \{\tau < \infty\}. \quad (\text{B.9})$$

c) There is a version of  $\nu$  such that  $\nu(\omega, \{t\} \times E) \leq 1$  identically, and the thin set  $J$  is exhausted by a sequence of predictable times.

*Remark B.12.* Because of the validity of property (B.9), the compensator  $\nu$  is also called the dual predictable projection of  $\mu$ .

**Proposition B.13.** Let  $\mu$  be an integer valued random measure with support  $D$ . Let  $J$  and  $K$  be the associated sets defined in (B.7) and (B.8). If  $D = K \cup (\cup_n [[S_n]])$ , where  $(S_n)_n$  are totally inaccessible times, then  $J = K$  up to an evanescent set.

*Proof.* We start by noticing some basic facts. From the definition of predictable support of a random set in Definition A.25, we have

$$\mathbb{1}_J = {}^p(\mathbb{1}_D). \quad (\text{B.10})$$

Moreover, since  $K$  is predictable, by Corollary A.24 we get

$${}^p(\mathbb{1}_K) = \mathbb{1}_K; \quad (\text{B.11})$$

on the other hand, by Remark A.27 the predictable projection of  $\mathbb{1}_{[[S_n]]}$  is zero since  $S_n$  is a totally inaccessible finite time. Consequently we obtain

$${}^p(\mathbb{1}_{\cup_n [[S_n]]}) = \sum_n {}^p(\mathbb{1}_{[[S_n]]}) = 0. \quad (\text{B.12})$$

Finally, identities (B.10), (B.11) and (B.12) imply

$$\mathbb{1}_J = {}^p(\mathbb{1}_D) = \mathbb{1}_K,$$

therefore  $J = K$ . □

### B.3 Stochastic integrals with respect to an integer-valued random measure.

From here on  $\mu$  will be an integer-valued random measure on  $[0, T] \times E$ , and  $\nu$  a "good" version of the compensator of  $\mu$  as constructed in Proposition B.11-(c).

We set  $\hat{\nu}_t(de) = \nu(\{t\}, de)$  for all  $t \in [0, T]$  and, for any  $W \in \tilde{\mathcal{O}}$ , we define

$$\hat{W}_t = \int_E W_t(x) \hat{\nu}_t(de), \quad t \geq 0, \quad (\text{B.13})$$

$$\tilde{W}_t = \int_E W_t(x) \mu(\{t\}, de) - \int_E W_t(x) \hat{\nu}_t(de) = W_t(\beta_t) \mathbb{1}_D(t) - \hat{W}_t, \quad t \geq 0, \quad (\text{B.14})$$

with the convention

$$\tilde{W}_t = +\infty \quad \text{if } \hat{W}_t \text{ is not defined.} \quad (\text{B.15})$$

$\beta$  and  $D$  in (B.14) are respectively the optional process and the support associated to  $\mu$ , see Proposition B.6. For every  $q \in [1, \infty[$ , we also introduce the following linear spaces

$$\mathcal{G}^q(\mu) = \left\{ W \in \tilde{\mathcal{P}} : \left[ \sum_{s \leq \cdot} |\tilde{W}_s|^2 \right]^{q/2} \in \mathcal{A}^+ \right\}, \quad (\text{B.16})$$

$$\mathcal{G}_{\text{loc}}^q(\mu) = \left\{ W \in \tilde{\mathcal{P}} : \left[ \sum_{s \leq \cdot} |\tilde{W}_s|^2 \right]^{q/2} \in \mathcal{A}_{\text{loc}}^+ \right\}. \quad (\text{B.17})$$

We have  $\mathcal{G}^q(\mu) \subset \mathcal{G}^{q'}(\mu)$  for every  $q' \leq q$ .

*Remark B.14.* The sets in (B.17) corresponding to  $q = 1, 2$  coincide respectively with the spaces  $\mathcal{G}(\mu)$  and  $\mathcal{G}^2(\mu)$  introduced in [18], pages 301 and 304. In particular, under convention (B.15), any element  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$  satisfies  $|\hat{W}_t| < \infty$  for every  $t \in [0, T]$ .

*Remark B.15.* If  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$ , then exists a local martingale  $M$  such that  $\Delta M$  and  $\tilde{W}$  are indistinguishable.

This is a consequence of the fact that the predictable projection of  $\tilde{W}$  is zero, see observations below Definition 1.27, Chapter II, in [21], and of Theorem A.28 with  $H = \tilde{W}$ .

**Definition B.16** (Definition 1.27, point b), Chapter II, in [21]). *If  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$ , we call stochastic integral of  $W$  with respect to  $\mu - \nu$  and  $W * (\mu - \nu)$  denotes any purely discontinuous local martingale  $M$  such that  $\Delta M$  and  $\tilde{W}$  are indistinguishable.*

*Remark B.17.* By Corollary A.9, if  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$ , all the stochastic integrals  $W * (\mu - \nu)$  are equal up to indistinguishability.

**Proposition B.18** (Proposition 1.28, Chapter II, in [21]). *Let  $W \in \tilde{\mathcal{P}}$ , such that  $|W| * \mu \in \mathcal{A}_{\text{loc}}^+$  (or equivalently, by Theorem B.4,  $|W| * \nu \in \mathcal{A}_{\text{loc}}^+$ ). Then  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$  and*

$$W * (\mu - \nu) = W * \mu - W * \nu.$$

For any  $W \in \tilde{\mathcal{P}}$ , let now define the following two increasing (possibly infinite) predictable processes

$$\begin{aligned} C(W)_t &= |W - \hat{W}|^2 * \nu_t + \sum_{s \leq t} (1 - \hat{\nu}_s(E)) |\hat{W}_s|^2, \\ \bar{C}(W)_t &= |W - \hat{W}| * \nu_t + \sum_{s \leq t} (1 - \hat{\nu}_s(E)) |\hat{W}_s|. \end{aligned} \quad (\text{B.18})$$

The sets  $\mathcal{G}_{\text{loc}}^1(\mu)$  and  $\mathcal{G}_{\text{loc}}^2(\mu)$  can be characterized in the following way.

**Theorem B.19** (Theorem 1.33, point c), Chapter II, in [21]). *Let  $W \in \tilde{\mathcal{P}}$ . Then  $W$  belongs to  $\mathcal{G}_{\text{loc}}^1(\mu)$  if and only if  $C(W') + \bar{C}(W'')$  belongs to  $\mathcal{A}_{\text{loc}}^+$ , where*

$$\begin{cases} W' = (W - \hat{W}) \mathbb{1}_{\{|W - \hat{W}| \leq 1\}} + \hat{W} \mathbb{1}_{\{|\hat{W}| \leq 1\}}, \\ W'' = (W - \hat{W}) \mathbb{1}_{\{|W - \hat{W}| > 1\}} + \hat{W} \mathbb{1}_{\{|\hat{W}| > 1\}}. \end{cases}$$

**Proposition B.20** (Proposition 3.71 in [20]). *Let  $W \in \tilde{\mathcal{P}}$ . Then  $W \in \mathcal{G}^2(\mu)$  if and only if  $C(W) \in \mathcal{A}^+$ .*

By Proposition B.20, the space  $\mathcal{G}^2(\mu)$  can be rewritten as

$$\mathcal{G}^2(\mu) = \{W \in \tilde{\mathcal{P}} : \|W\|_{\mathcal{G}^2(\mu)} < \infty\},$$

where

$$\|W\|_{\mathcal{G}^2(\mu)}^2 := \mathbb{E}[C(W)] = \mathbb{E}\left[\int_{[0, T] \times E} |W_s(e) - \hat{W}_s|^2 \nu(ds de) + \sum_{s \leq T} |\hat{W}_s|^2 (1 - \hat{\nu}_s(E))\right]. \quad (\text{B.19})$$

Let us introduce the space

$$\mathcal{L}^2(\mu) := \{W \in \tilde{\mathcal{P}} : \|W\|_{\mathcal{L}^2(\mu)} < \infty\} \quad (\text{B.20})$$

with

$$\|W\|_{\mathcal{L}^2(\mu)} := \mathbb{E} \left[ \int_{[0,T] \times \mathbb{R}} |W_s(e)|^2 \nu(ds de) \right].$$

We have the following result.

**Lemma B.21.** 1. If  $W \in \mathcal{L}^2(\mu)$ , then  $W \in \mathcal{G}^2(\mu)$  and

$$\|W\|_{\mathcal{G}^2(\mu)}^2 \leq \|W\|_{\mathcal{L}^2(\mu)}^2. \quad (\text{B.21})$$

2. If  $|W|^2 * \mu \in \mathcal{A}_{\text{loc}}^+$  then  $W \in \mathcal{G}_{\text{loc}}^2(\mu)$ .

*Proof.* Let  $W \in \tilde{\mathcal{P}}$ . For every  $t \geq 0$ , since  $\hat{\nu}_t(\mathbb{R}) \leq 1$ , we have

$$\sum_{s \in ]0, t]} |\hat{W}_s|^2 (1 - \hat{\nu}_s(E)) \leq \sum_{s \leq t} |\hat{W}_s|^2 \leq \sum_{s \leq t} \hat{\nu}_s(E) \int_E |W_s(e)|^2 \hat{\nu}_s(de) \leq |W|^2 * \nu_t. \quad (\text{B.22})$$

Assume now that moreover  $W \in \mathcal{L}^2(\mu)$ . Then (B.22), together with the triangle inequality, implies that

$$\mathbb{E} \left[ \sum_{s \in ]0, T]} |\hat{W}_s|^2 (1 - \hat{\nu}_s(E)) \right] < \infty, \quad \mathbb{E} \left[ \int_{[0, T] \times E} |W_s(e) - \hat{W}_s|^2 \nu(ds de) \right] < \infty,$$

i.e.,  $W \in \mathcal{G}^2(\mu)$ . Moreover, taking into account that

$$|\hat{W}|^2 * \nu_t = \sum_{s \leq t} |\hat{W}_s|^2 \hat{\nu}_s(E), \quad \forall t \geq 0, \quad (\text{B.23})$$

the process  $C(W)$  defined in (B.18) can be decomposed as

$$\begin{aligned} C(W)_t &= |W|^2 * \nu_t - 2 \sum_{s \leq t} |\hat{W}_s|^2 + \sum_{s \leq t} |\hat{W}_s|^2 \hat{\nu}_s(E) + \sum_{s \leq t} |\hat{W}_s|^2 (1 - \hat{\nu}_s(E)) \\ &= |W|^2 * \nu_t - \sum_{s \leq t} |\hat{W}_s|^2. \end{aligned} \quad (\text{B.24})$$

In particular, we have

$$\|W\|_{\mathcal{G}^2(\mu)}^2 = \mathbb{E} \left[ \int_{[0, T] \times \mathbb{R}} |W_s(e)|^2 \nu(ds de) - \sum_{s \in ]0, T]} |\hat{W}_s|^2 \right] \leq \|W\|_{\mathcal{L}^2(\mu)}^2.$$

This establishes point 1. Point 2. follows by usual localization arguments.  $\square$

**Theorem B.22** (Theorem 11.21, point 3), in [18]). *Let  $W \in \tilde{\mathcal{P}}$ . The following properties are equivalent.*

- (i)  $W$  belongs to  $\mathcal{G}_{\text{loc}}^2(\mu)$ .
- (ii)  $C(W)$  belongs to  $\mathcal{A}_{\text{loc}}^+$ .
- (iii)  $W$  belongs to  $\mathcal{G}_{\text{loc}}^1(\mu)$  and  $W * (\mu - \nu)$  belongs to  $\mathcal{H}_{\text{loc}}^{2,d}$ .

In this case, we have

$$\langle W * (\mu - \nu), W * (\mu - \nu) \rangle_t = C(W)_t. \quad (\text{B.25})$$

If in addition  $|W|^2 * \nu_t \in \mathcal{A}_{\text{loc}}^+$ , then

$$\langle W * (\mu - \nu), W * (\mu - \nu) \rangle_t = |W|^2 * \nu_t - \sum_{s \leq t} |\hat{W}_s|^2. \quad (\text{B.26})$$

*Remark B.23.* Let  $W \in \tilde{\mathcal{P}}$ , and  $\mu$  an integer-valued random measure with support  $D$ . We recall that the random sets  $J$  and  $K$  have been introduced in Theorem B.10. By definition of  $\hat{W}$ ,  $J$  and  $K$ . We have

$$\hat{W} = \hat{W} \mathbb{1}_J, \quad (\text{B.27})$$

$$\hat{\nu}(E) \mathbb{1}_K = \mathbb{1}_K, \quad (\text{B.28})$$

$$1 - \hat{\nu}(E) > 0 \quad \text{on } J \setminus K. \quad (\text{B.29})$$

Taking into account (B.27), (B.28) and (B.29), we see that the quantity  $C(W)$  in (B.18) can be rewritten as

$$C(W) = |W - \hat{W} \mathbb{1}_J|^2 * \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) |\hat{W}_s|^2 \mathbb{1}_{J \setminus K}(s). \quad (\text{B.30})$$

In the particular case of  $K = J$ , previous identity reduces to

$$C(W) = |W - \hat{W} \mathbb{1}_K|^2 * \nu. \quad (\text{B.31})$$

Denoting  $\nu^d = \nu \mathbb{1}_K$  and  $\nu^c = \nu \mathbb{1}_{K^c}$ , then

$$C(W) = \int_{]0, \cdot] \times \mathbb{R}} |W_s(e)|^2 \nu^c(ds de) + \int_{]0, \cdot] \times \mathbb{R}} |W_s(e) - \hat{W}_s \mathbb{1}_K(s)|^2 \nu^d(ds, de). \quad (\text{B.32})$$

*Remark B.24.* It directly follows from (B.31) and from the definition of the  $\mathcal{G}^2(\mu)$  seminorm (see (B.19)) that if  $K = J$ , then

$$\|W\|_{\mathcal{G}^2(\mu)}^2 = \|W - \hat{W} \mathbb{1}_K\|_{\mathcal{L}^2(\mu)}^2 = \|W - \hat{W}\|_{\mathcal{L}^2(\mu)}^2.$$

**Proposition B.25.** *Let  $(l_s)$  be a predictable process. Then  $C(l \mathbb{1}_K) = 0$ .*

*Proof.* By definition

$$(\widehat{l_s \mathbb{1}_K(s)}) = \int_E l_s \mathbb{1}_K(s) \hat{\nu}_s(de) = l_s \mathbb{1}_K(s) \hat{\nu}_s(E) = l_s \mathbb{1}_K(s), \quad (\text{B.33})$$

where the latter equality follows from (B.28). Then (B.30) in Remark B.23 gives

$$C(l \mathbb{1}_K) = |l \mathbb{1}_K - l \mathbb{1}_K|^2 * \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) \mathbb{1}_K(s) |l_s|^2 \mathbb{1}_{J \setminus K}(s) = 0.$$

□

**Proposition B.26.** *Let  $W \in \tilde{\mathcal{P}}$ . Then for any predictable process  $(l_s)$ ,*

$$C(W) = C(W + l \mathbb{1}_K).$$

*Proof.* We designate  $W^0 = W + l \mathbb{1}_K$ . Taking into account (B.33), we have

$$\hat{W}_s^0 = (\widehat{W_s + l_s \mathbb{1}_K}(s)) = \hat{W}_s + l_s \mathbb{1}_K(s).$$

Then, recalling (B.30), we get

$$\begin{aligned} C(W^0) &= |W^0 - \hat{W}^0 \mathbb{1}_J|^2 * \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) |\hat{W}_s^0|^2 \mathbb{1}_{J \setminus K}(s) \\ &= |W + l \mathbb{1}_K - \hat{W} \mathbb{1}_J - l \mathbb{1}_K|^2 * \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) |\hat{W}_s + l_s \mathbb{1}_K(s)|^2 \mathbb{1}_{J \setminus K}(s) \\ &= |W - \hat{W} \mathbb{1}_J|^2 * \nu + \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) |\hat{W}_s|^2 \mathbb{1}_{J \setminus K}(s) = C(W). \end{aligned}$$

□

**Corollary B.27.** *Let  $(l_s)_{s \in [0, T]}$  be a predictable process. If  $W \in \mathcal{G}^2(\mu)$ , then*

$$W + l \mathbb{1}_K \in \mathcal{G}^2(\mu), \quad (\text{B.34})$$

and

$$\|W + l \mathbb{1}_K\|_{\mathcal{G}^2(\mu)} = \|W\|_{\mathcal{G}^2(\mu)}. \quad (\text{B.35})$$

*Proof.* (B.34) (resp. (B.35)) is a consequence of Proposition B.26 and Proposition B.20 (resp. formula (B.19)). □

**Proposition B.28.** *If  $W \in \mathcal{G}^2(\mu)$  and  $\|W\|_{\mathcal{G}^2(\mu)} = 0$ , then*

$$\|W - \hat{W} \mathbb{1}_K\|_{\mathcal{L}^2(\mu)} = 0. \quad (\text{B.36})$$

*In particular, there is a predictable process  $(l_s)$  such that*

$$W_s(e) = l_s \mathbb{1}_K(s), \quad \nu(ds de)\text{-a.e.}$$

*Proof.* Since  $\|W\|_{\mathcal{G}^2(\mu)} = 0$ , we have  $C(W)_T = 0$  a.s., see (B.19). Recalling (B.30), this implies

$$\begin{cases} |W - \hat{W} \mathbb{1}_J|^2 * \nu = 0, \\ \sum_{s \leq \cdot} (1 - \hat{\nu}_s(E)) |\hat{W}_s|^2 \mathbb{1}_{J \setminus K}(s) = 0. \end{cases}$$

Since  $1 - \hat{\nu}(E) > 0$  on  $J \setminus K$  (see Remark B.23), previous identities imply

$$\begin{cases} |W - \hat{W} \mathbb{1}_J|^2 * \nu = 0, \\ \hat{W} \mathbb{1}_{J \setminus K} = 0, \end{cases}$$

which gives (B.36). □

*Remark B.29.* If  $K = \emptyset$ , then

$$\|W\|_{\mathcal{G}^2(\mu)}^2 = 0 \text{ if and only if } \|W\|_{\mathcal{L}^2(\mu)}^2 = 0.$$

Indeed, by Proposition B.28,  $K = \emptyset$  and  $\|W\|_{\mathcal{G}^2(\mu)}^2 = 0$  imply that  $\|W\|_{\mathcal{L}^2(\mu)}^2 = 0$ . The opposite implication follows from the fact that  $\|W\|_{\mathcal{G}^2(\mu)}^2 \leq \|W\|_{\mathcal{L}^2(\mu)}^2$ , see Lemma B.21.

We end this section with an important result of the stochastic integration theory.

**Proposition B.30.** *Let  $W \in \mathcal{G}_{\text{loc}}^1(\mu)$ , and define  $M_t = \int_{]0,t] \times \mathbb{R}} W_s(e) (\mu - \nu)(ds de)$ . Let moreover  $(Z_t)$  be a predictable process such that*

$$\sqrt{\sum_{s \leq \cdot} Z_s^2 |\Delta M_s|^2} \in \mathcal{A}_{\text{loc}}^+. \quad (\text{B.37})$$

Then  $\int_0^\cdot Z_s dM_s$  is a local martingale and equals

$$\int_{]0,\cdot] \times \mathbb{R}} Z_s W_s(e) (\mu - \nu)(ds de). \quad (\text{B.38})$$

*Remark B.31.* Since  $M$  is a local martingale,  $\sqrt{[M, M]_t} \in \mathcal{A}_{\text{loc}}^+$ , see e.g. Theorem 2.34 and Proposition 2.38 in [20]. Taking into account that  $M$  is a purely jump local martingale, by Proposition 5.3 in [2] this is equivalent to  $\sqrt{\sum_{s \leq \cdot} |\Delta M_s|^2} \in \mathcal{A}_{\text{loc}}^+$ . Then condition (B.37) is verified if for instance when  $Z$  is locally bounded.

*Proof.* The conclusion follows by the definition of the stochastic integral (B.38), see Definition B.16, provided we check the following three conditions.

- (i)  $\int_0^\cdot Z_s dM_s$  is a local martingale.
- (ii)  $\int_0^\cdot Z_s dM_s$  is a purely discontinuous local martingale; in agreement with Theorem A.6, we will show  $[\int_0^\cdot Z_s dM_s, N] = 0$  for every  $N$  continuous local martingale vanishing at zero.
- (iii)  $\Delta \left( \int_0^\cdot Z_s dM_s \right)_t = \int_{\mathbb{R}} Z_t W_t(e) (\mu(\{t\}, de) - \nu(\{t\}, de)), \quad t \in [0, T].$

We prove now the validity of (i), (ii) and (iii).

Condition (B.37) is equivalent to  $\sqrt{\int_0^t Z_s^2 d[M, M]_s} \in \mathcal{A}_{\text{loc}}^+$ . According to Definition 2.46 in [20],  $\int_0^t Z_s dM_s$  is the unique local martingale satisfying

$$\Delta \left( \int_0^\cdot Z_s dM_s \right)_t = Z_t \Delta M_t, \quad t \in [0, T]. \quad (\text{B.39})$$

This implies in particular item (i).

By Theorem 29, Chapter II, in [29], it follows that

$$\left[ \int_0^\cdot Z_s dM_s, N \right] = \int_0^\cdot Z_s d[M, N]_s,$$

and item (ii) follows because  $M$  is orthogonal to  $N$ , see Theorem A.6.

Finally, by Definition B.16, taking into account (B.39),  $\Delta \left( \int_0^\cdot Z_s dM_s \right)_t$  equals

$$Z_t \Delta M_t = \int_{\mathbb{R}} Z_t W_t(e) (\mu(\{t\}, de) - \nu(\{t\}, de))$$

for every  $t \in [0, T]$ , and this shows item (iii).  $\square$

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